# ASW vs. WHDG

comparison

Q.: Does the kinematic limit $k_\perp < 2x(1-x)E$ implemented in WHDG violate the mass-shell constraint $k \cdot k = 0$ for the emitted gluon?

Notes by U. Heinz 5/18/2009

with help from Will Horowitz in solving the cubic equation on page 3

$$p = (p^0, p^z, \vec{p}_\perp) \equiv [p^+, p^-, \vec{p}_\perp]$$

$$x = (x^0, z, \vec{r}_\perp) \equiv [x^+, x^-, \vec{r}_\perp]$$

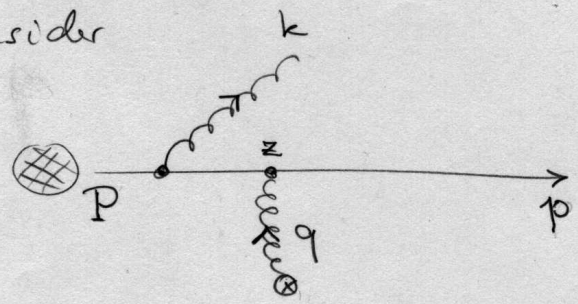
$$\begin{cases} p^+ = p^0 + p^z \\ p^- = p^0 - p^z \end{cases}$$

$$\begin{cases} p^0 = \frac{1}{2}(p^+ + p^-) \\ p^z = \frac{1}{2}(p^+ - p^-) \end{cases}$$

$$p \cdot k = p^0 k^0 - p^z k^z - \vec{p}_\perp \cdot \vec{k}_\perp$$

$$= \frac{1}{2}(p^+ k^- + p^- k^+) - \vec{p}_\perp \cdot \vec{k}_\perp$$

Consider




- fast parton $p$ is massless $\boxed{p \cdot p = 0}$
- emitted gluon $k$ is massless $\boxed{k \cdot k = 0}$
- scattering center $\otimes$ is fixed (infinitely massive) $\rightarrow q = (0, q^z, \vec{q}_\perp)$

**ASW:**

$$P = (E, E, 0, 0) \qquad (E^+ = 2E)$$

$$k = (\omega, k_L, \vec{k}_\perp)$$

$$\equiv (xE, k_L, k_\perp)$$

$$k_L = \sqrt{(xE)^2 - k_\perp^2}$$

$$p = ((1-x)E, p_L, \vec{q}_\perp - \vec{k}_\perp)$$

$$p_L = \sqrt{(1-x)^2 E^2 - (\vec{q}_\perp - \vec{k}_\perp)^2}$$

$$x = \frac{\omega}{E}$$

$$= \tilde{x} + \frac{1}{\tilde{x}} \frac{k_\perp^2}{4E^2}$$

**WHDG:**

$$P = [E^+, 0, 0, 0]$$

$$k = \left[\tilde{x} E^+, \frac{k_\perp^2}{\tilde{x} E^+}, \vec{k}_\perp\right]$$

$$\omega = k^0 = \frac{1}{2}(k^+ + k^-) =$$

$$= \frac{(\tilde{x} E^+)^2 + k_\perp^2}{2 \tilde{x} E^+} = \frac{4\tilde{x}^2 E^2 + k_\perp^2}{4 \tilde{x} E}$$

$$= \tilde{x} E + \frac{k_\perp^2}{4 \tilde{x} E}$$

$$p = \left[(1-\tilde{x}) E^+, \frac{(\vec{q}_\perp - \vec{k}_\perp)^2}{(1-\tilde{x}) E^+}, \vec{q}_\perp - \vec{k}_\perp\right]$$

$$\tilde{x} = \frac{k^+}{E^+} = x\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{k_\perp^2}{(xE)^2}}\right)$$

$$\approx x - \frac{1}{x} \frac{k_\perp^2}{4E^2} \quad \left(\text{if } \frac{k_\perp^2}{(xE)^2} \ll 1\right)$$

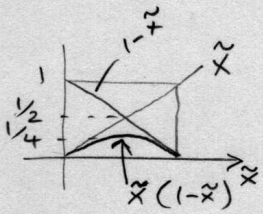
## ASW

$$0 \le k^0 \le E$$

$$\Rightarrow 0 \le x \le 1$$

$$k \cdot k = 0 \Rightarrow \omega = \sqrt{k_L^2 + \vec{k}_\perp^2} \ge k_\perp$$

$$\Rightarrow \boxed{k_\perp < \omega = xE}$$




$\longrightarrow$

At the limit $k_\perp = xE$

$$\tilde{x} = x\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{k_\perp^2}{(xE)^2}}\right) = \frac{x}{2}$$

$\Rightarrow$ At the limit $k_\perp = xE$, $\tilde{x} < x$ !

Q:

## WHDG

$$0 \le k^+ \le E^+$$

$$\Rightarrow 0 \le \tilde{x} \le 1$$

$$k_L = \frac{1}{2}(k^+ - k^-) > 0 \Rightarrow$$

$$\tilde{x}^2 E^{+2} - k_\perp^2 > 0 \Rightarrow \boxed{k_\perp < \tilde{x} E^+}$$

$$p_L > 0 \Rightarrow (1-\tilde{x})^2 E^{+2} - (\vec{q}_\perp - \vec{k}_\perp)^2 > 0$$

$$\Longrightarrow |\vec{k}_\perp - \vec{q}_\perp| < (1-\tilde{x}) E^+$$

WHDG assume here that

$|\vec{k}_\perp - \vec{q}_\perp| \approx k_\perp$, i.e. $|\vec{q}_\perp| \ll |\vec{k}_\perp|$ (?)

$$\Rightarrow k_\perp < \min(\tilde{x}, 1-\tilde{x}) E^+$$

$$= 2E \min(\tilde{x}, 1-\tilde{x}) \approx 2\tilde{x}(1-\tilde{x})E$$

At the limit $k_\perp = 2\tilde{x}(1-\tilde{x})E$

$$x = \tilde{x} + \frac{k_\perp^2}{4\tilde{x}E^2} =$$

$$= \tilde{x} + \frac{4\tilde{x}^2(1-\tilde{x})^2 E^2}{4\tilde{x}E^2}$$

$$= \tilde{x} + \tilde{x}(1-\tilde{x})^2 = \tilde{x}(1 + (1-\tilde{x})^2)$$

At the limit $k_\perp = 2\tilde{x}(1-\tilde{x})E$, $x > \tilde{x}$ !

At the WHDG kinematic limit we have

$k_\perp^{max} = 2\tilde{x}(1-\tilde{x})E$ which can be expressed through $x$ as

$$\frac{k_\perp^{max}}{E} = x\left(1+\sqrt{1-\frac{k_{\perp,max}^2}{(xE)^2}}\right)\left(1-\frac{x}{2}\left(1+\sqrt{1-\frac{k_{\perp,max}^2}{(xE)^2}}\right)\right)$$

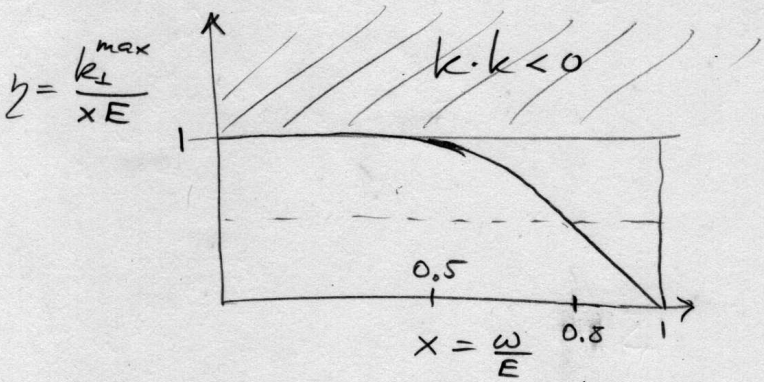
Use the definition of $x$, $\omega = xE$, and define $\eta = \frac{k_\perp^{max}}{xE} = \frac{k_\perp^{max}}{\omega}$

For $k$ to be on-shell, we need $\eta \leq 1$.

$$\boxed{\eta = \left(1+\sqrt{1-\eta^2}\right)\left(1-\frac{x}{2}\left(1+\sqrt{1-\eta^2}\right)\right)} \longrightarrow \eta(x)$$

(cubic equation)

Mathematica gives the following for $\eta(x)$ (see attached plot)




So the WHDG kinematic constraint is, in fact <u>more</u> restrictive that the ASW constraint $k_\perp < \omega = xE$ !

## ■ Mixed Solns

In[16]:=

$$\text{Plot}\Big[\Big\{\text{Re}\Big[\frac{4}{3x}+\frac{\left(1+i\sqrt{3}\right)\left(8x^2-12x^3\right)}{6\,2^{2/3}x^2\left(-20x^3+45x^4-27x^5+3\sqrt{3}\sqrt{16x^6-72x^7+123x^8-94x^9+27x^{10}}\right)^{1/3}}-\frac{\left(1-i\sqrt{3}\right)\left(-20x^3+45x^4-27x^5+3\sqrt{3}\sqrt{16x^6-72x^7+123x^8-94x^9+27x^{10}}\right)^{1/3}}{3\,2^{1/3}x^2}\Big],$$

$$\text{Re}\Big[\frac{4}{3x}-\frac{8x^2-12x^3}{3\,2^{2/3}x^2\left(-20x^3+45x^4-27x^5+3\sqrt{3}\sqrt{16x^6-72x^7+123x^8-94x^9+27x^{10}}\right)^{1/3}}+\frac{2^{2/3}\left(-20x^3+45x^4-27x^5+3\sqrt{3}\sqrt{16x^6-72x^7+123x^8-94x^9+27x^{10}}\right)^{1/3}}{3x^2}\Big]\Big\},$$

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{x, 0, 1}, PlotStyle → {{Red, Thick}, {Blue, Thick}}, Frame → True,
Axes → False, PlotRange → {0, 2}]
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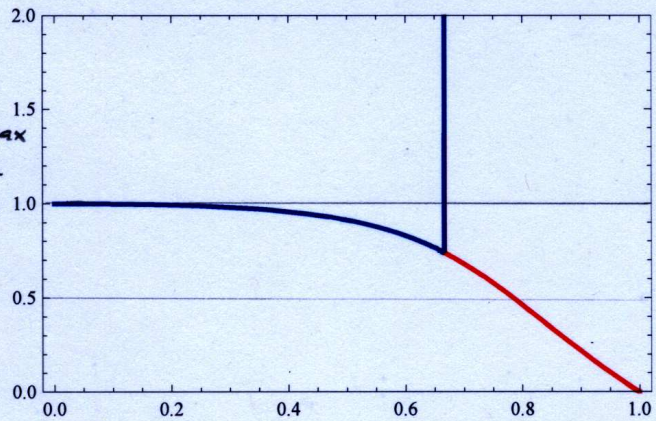
$\eta = \frac{k_\perp^{max}}{\omega}$

Out[16]=




$x = \frac{\omega}{E}$

Mathematica gives 3 solutions; one is ~~pure~~ complex for all $x$, one (blue) is purely real for $x < 0.65\ldots$, the third (red) is purely real for $x > 0.65\ldots$ (the vertical line indicates the "cross-over" point)

$x > 0.8$, $\frac{k_T}{\omega} \ll 1$ whereas

$\frac{k_T^{max,A}}{\omega} = 1$. This suggests that at large $x > 0.8$, near the ASW kinematic limit $k_T^{max,A} = \omega = xE$, one or both of the conditions $k_L > 0$, $p_L > 0$ imposed by WHDG is violated.

- Since $\omega^2 = k_L^2 + k_\perp^2 \Rightarrow k_L^2 = \omega^2 - k_\perp^2$, so if $k_L > 0$ at $k_\perp = 0$, it remains positive in the entire region $0 < k_\perp < k_\perp^{max,A} = \omega$, so there is no problem with violating $k_L > 0$ by going with $k_\perp$ beyond the WHDG limit $k_\perp^{max,W} = 2\tilde{x}(1-\tilde{x})E$.

- What about $p_L$? From p. 1 we have

$$p_L = \sqrt{(1-x)^2E^2 - (\vec{q}_\perp - \vec{k}_\perp)^2} \Rightarrow p_L^2 = (1-x)^2E^2 - (\vec{q}_\perp - \vec{k}_\perp)^2$$
$$\approx (1-x)^2E^2 - k_\perp^2$$

(assuming, as WHDG do, that $k_\perp \gg \sqrt{\langle q_\perp^2 \rangle} \sim O(T)$, i.e. that the emitted gluon has transverse momentum larger than the typical $q_\perp$ in the "medium structure function")

Check where $p_L$ becomes zero:

$$k_\perp = (1-x)E$$

$$\boxed{\text{So if } x > 1/2 \text{ and } k_\perp > \tfrac{1}{2}\omega, \text{ then } p_L^2 \text{ is negative!}}$$

- So the stricter WHDG kinematic limit is necessary to

on-shell condition for the outgoing fast parton to be violated.

This suggests to me that it is <u>important</u> to restrict the $k_\perp$ integration domain to the tighter WHDG limit $\boxed{k_\perp < 2\tilde{x}(1-\tilde{x})E} < xE$, and that this more restrictive limit influences in particular gluon radiation where the radiated gluon takes a large fraction of the initial energy of the hard parton away.