Viscous hydrodynamics and covariant transport

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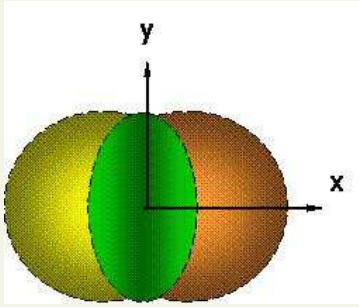
- **Viscosity, causal viscous hydrodynamics (Israel-Stewart)**
- **Region of validity for viscous hydrodynamics, shear viscosity at RHIC**
- **Implications of bulk viscosity**

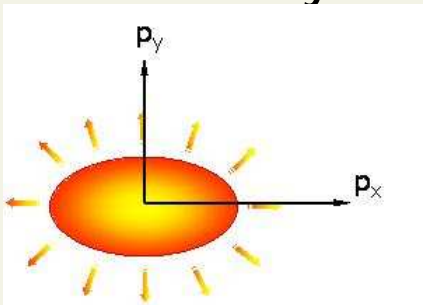
in collaboration with Pasi Huovinen

Thermalization at RHIC

efficient conversion of spatial eccentricity to momentum anisotropy



$\longrightarrow$

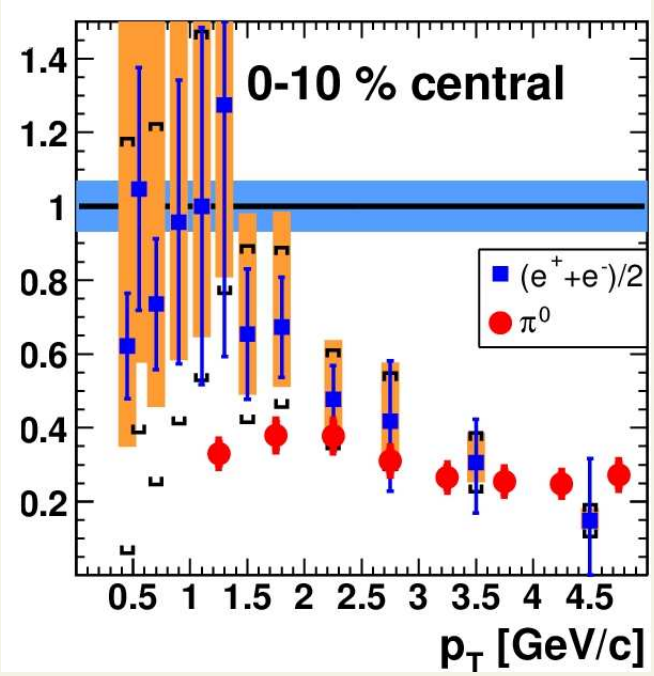


"elliptic flow"

$$\varepsilon \equiv \frac{\langle x^2 - y^2 \rangle}{\langle x^2 + y^2 \rangle} \qquad v_2 \equiv \frac{\langle p_x^2 - p_y^2 \rangle}{\langle p_x^2 + p_y^2 \rangle} \equiv \langle \cos 2\phi_p \rangle$$

very opaque - large energy loss, even for heavy quarks

$$R_{AA} = \frac{measured\ yield}{expected\ yield\ for\ dilute\ system}$$



Ideal (nondissipative) hydrodynamics describes particle spectra and elliptic flow surprisingly well Kolb & Heinz, QGP Vol. 3 [nucl-th/0305084], Kolb, Heinz, Huovinen et al ('01), ...

Hydrodynamics

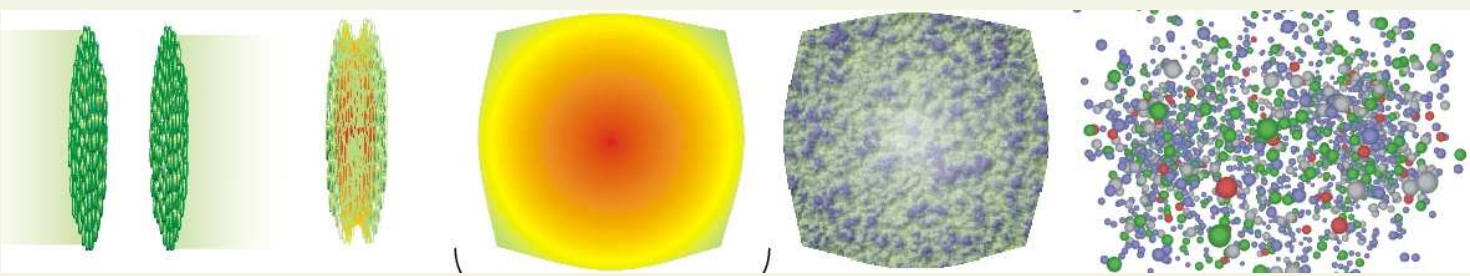
- **describes a system near local equilibrium**

- **long-wavelength, long-timescale dynamics, driven by conservation laws**

$$\partial_\mu T^{\mu\nu}(x) = 0, \qquad \partial_\mu N_B^\mu = 0$$

- **mainly used for the plasma stage of the collision**

initial nuclei parton plasma hadronization hadron gas



$\uparrow$
$\sim 10^{-14}$ **m**
= 10 fermis
$\downarrow$

<—— $\sim 10^{-23}$ **sec = 10 yochto(!)-secs** ——>

amazing that hydrodynamics can be applicable at such microscopic scales

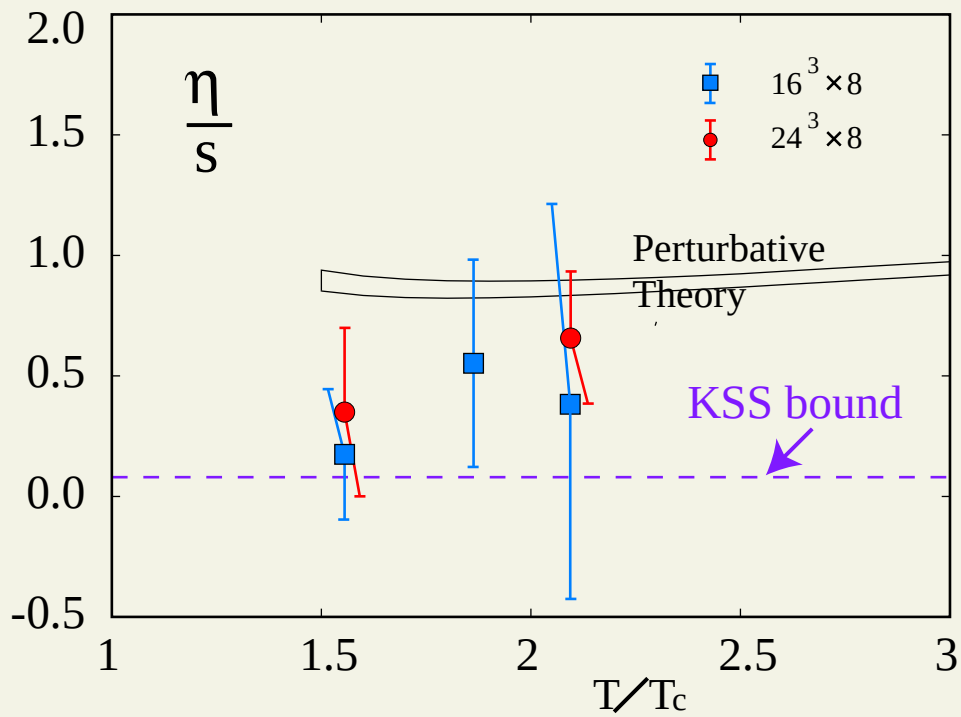
Shear viscosity in QCD

shear viscosity: $T_{xy} = -\eta \frac{\partial v_x}{\partial y}$ **acts to reduce velocity gradients**

largely unknown at $T \sim 200 - 400$ **MeV relevant for RHIC**

perturbatively: $\eta/s \sim \mathcal{O}(1)$, **lattice QCD: very preliminary**

Nakamura & Sakai, NPA774, 775 ('06):



Meyer, PRD76, 101701 ('07)

upper bounds:

$\eta/s(T{=}1.65T_c) < 0.96$

$\eta/s(T{=}1.24T_c) < 1.08$

- **no quarks (gluons only)**
- **crude lattices**

$\mathcal{N} = 4$ **super Yang-Mills (≠ QCD):** $\eta/s \geq \hbar/4\pi$

Policastro, Son, Starinets, PRL87 ('02)
Kovtun, Son, Starinets, PRL94 ('05)

In heavy-ion collisions, timescales are short, gradients are large - corrections to ideal (Euler) hydrodynamics are relevant.

Two ways to study dissipative effects:

- causal dissipative hydrodynamics

Israel, Stewart; ... Muronga, Rischke; Teaney et al; Romatschke et al; Heinz et al, DM & Huovinen

flexible in macroscopic properties

numerically cheaper

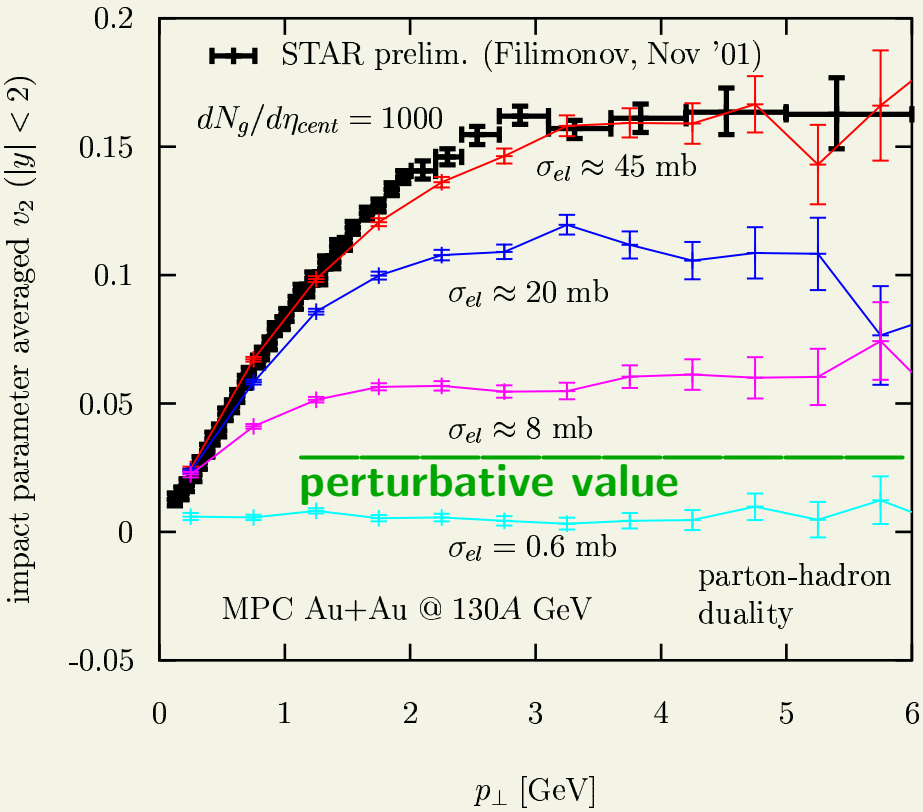
- covariant transport

Israel, de Groot,... Zhang, Gyulassy, DM, Pratt, Xu, Greiner...

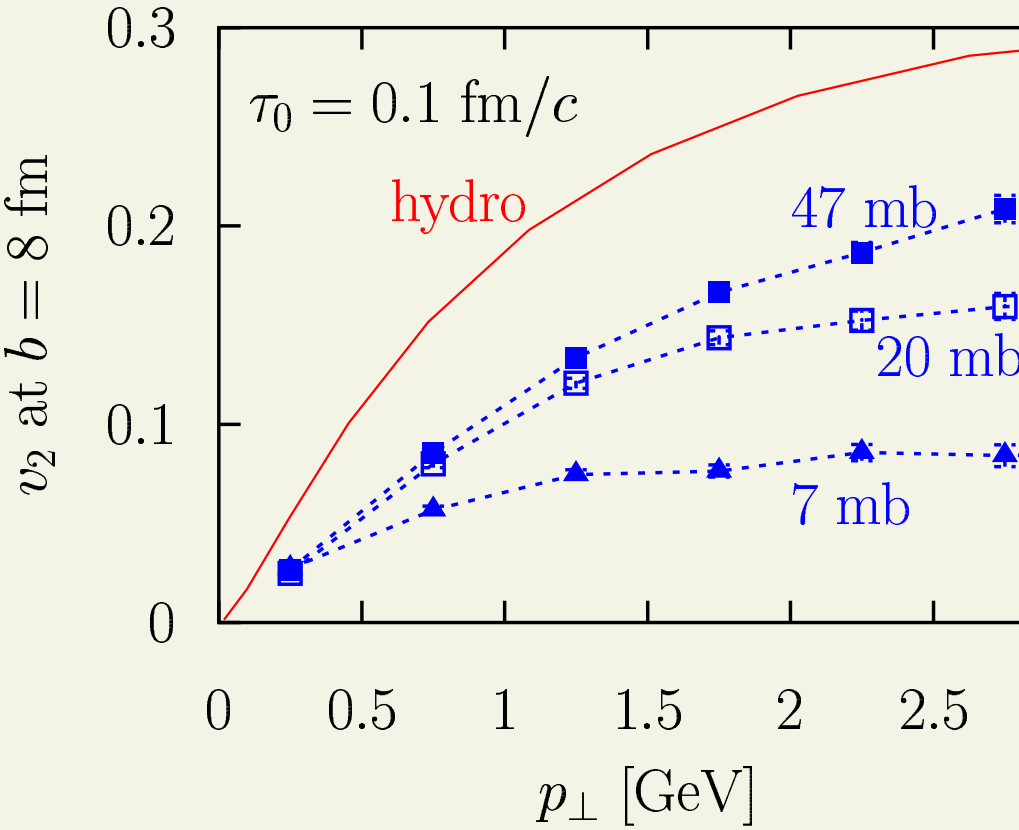
completely causal and stable

fully nonequilibrium $\longrightarrow$ interpolation to break-up stage

DM & Gyulassy, NPA 697 ('02): $v_2(p_T, \chi)$



DM & Huovinen, PRL94 ('05): **hydro vs transport**

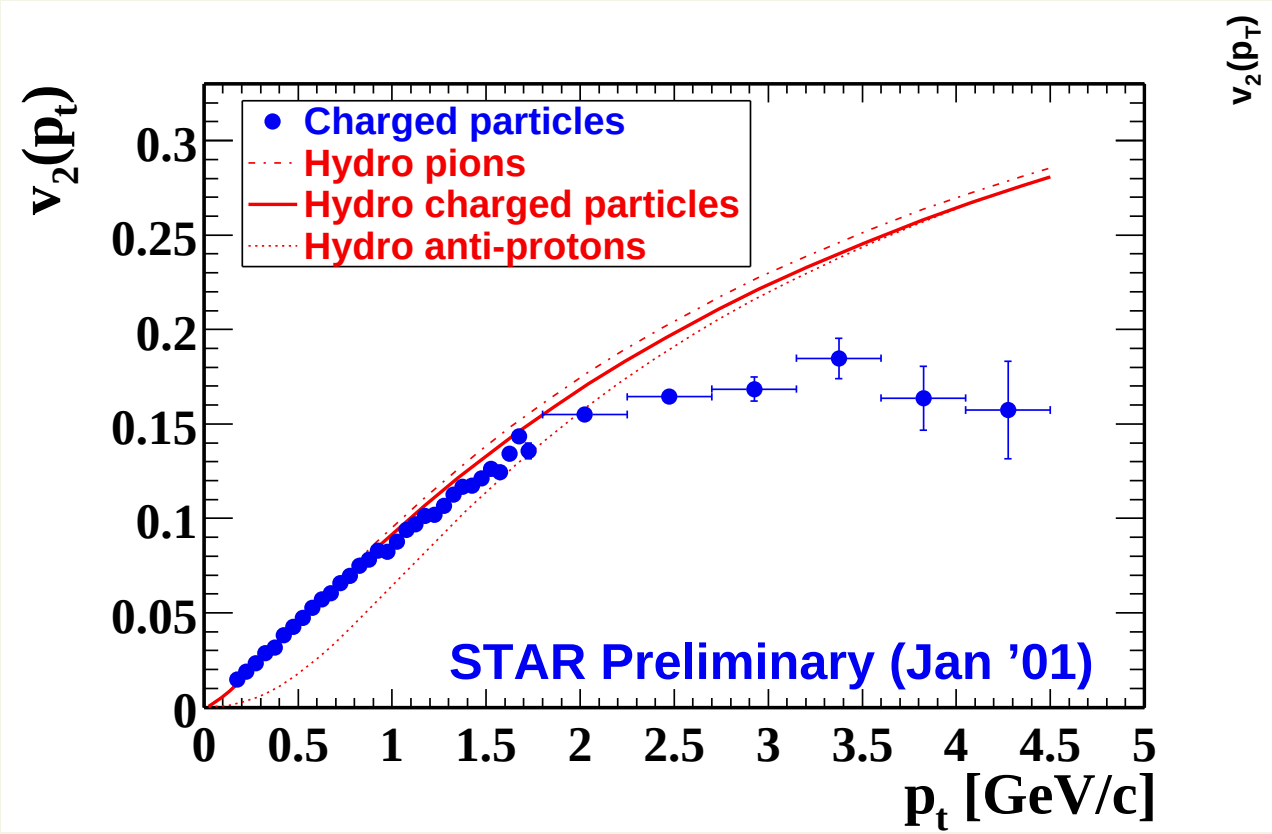


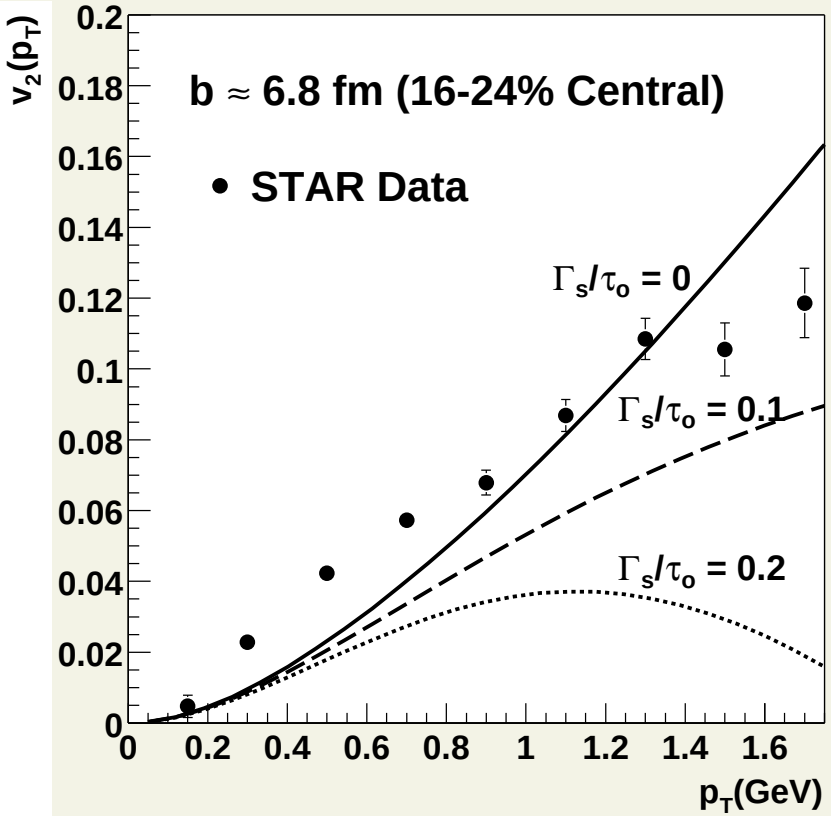
need $15\times$ perturbative $2 \to 2$ opacities to reproduce elliptic flow at RHIC

but even at such opacities, $20 - 30\%$ dissipative corrections

Snellings & Kolb et al, NPA 698, ('02):

Teaney ('04): v_2 **vs** $\Gamma_s \equiv \frac{4\eta}{3sT}$





ideal hydro overshoots elliptic flow at high pT

but viscous corrections to phase space distributions at freezeout help $\delta f \propto p^\mu p^\nu \nabla_\mu u_\nu$

Dissipative hydrodynamics

relativistic Navier-Stokes hydro: small corrections linear in gradients [Landau]

$$T^{\mu\nu}_{NS} = T^{\mu\nu}_{ideal} + \eta(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\partial^\alpha u_\alpha) + \zeta\Delta^{\mu\nu}\partial^\alpha u_\alpha$$

$$N^\nu_{NS} = N^\nu_{ideal} + \kappa\left(\frac{n}{\varepsilon + p}\right)^2 \nabla^\nu\left(\frac{\mu}{T}\right)$$

where $\Delta^{\mu\nu} \equiv u^\mu u^\nu - g^{\mu\nu}$, $\nabla^\mu = \Delta^{\mu\nu}\partial_\nu$ $[\partial^\mu \equiv u^\mu D + \nabla^\mu]$

η, ζ **shear and bulk viscosities,** κ **heat conductivity**

Equation of motion: $\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu N^\mu = 0$

two problems:

parabolic equations (e.g., heat flow) $\rightarrow$ **acausal** Müller ('76), Israel & Stewart ('79) ...

instabilities Hiscock & Lindblom, PRD31, 725 (1985) ...

Causal dissipative hydro

Bulk pressure Π, shear stress $\pi^{\mu\nu}$, heat flow q^μ are dynamical quantities

$$T^{\mu\nu} \equiv T^{\mu\nu}_{ideal} + \pi^{\mu\nu} - \Pi\Delta^{\mu\nu} \ , \qquad N^\mu \equiv N^\mu_{ideal} - \frac{n}{e+p} q^\mu$$

Israel-Stewart theory [Ann.Phys 100 & 118]**: relaxation equations**

$$\dot{X} = -\frac{1}{\tau_X}(X - X_{NS}) + X\,Y_X + Z_X$$

$\Rightarrow$ **alleviates the causality problem ($\tau_\Pi = \zeta\beta_0$, $\tau_q = \kappa_q T\beta_1$, $\tau_\pi = 2\eta_s\beta_2$)**

—

There may be more terms - e.g., imposing ONLY conformal invariance Baier, Romatschke, Son, JHEP04, 100 ('08)

$$\dot{\pi}^{\mu\nu} = \cdots + \frac{\lambda_1}{\eta^2}\pi^{\alpha\langle\mu}\pi_\alpha{}^{\nu\rangle} - \frac{\lambda_2}{\eta}\pi^{\alpha\langle\mu}\omega^{\nu\rangle}_\alpha + \lambda_3\omega^{\alpha\langle\mu}\omega_\alpha{}^{\nu\rangle} \tag{1}$$

No problem, we can solve with these too, provided $\lambda_{1,2,3}$ are known.

Israel-Stewart theory

starting point - truncate entropy current at quadratic order [Ann.Phys 100 & 118]

$$S^\mu = u^\mu \left[s - \frac{1}{2T} \left(\beta_0 \Pi^2 - \beta_1 q_\nu q^\nu + \beta_2 \pi_{\nu\lambda} \pi^{\nu\lambda} \right) \right] + \frac{q^\mu}{T} \left(\frac{\mu n}{\varepsilon + p} + \alpha_0 \Pi \right) - \frac{\alpha_1}{T} \pi^{\mu\nu} q_\nu$$

in Landau frame ($\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = \beta_2 = 0$ **gives Navier-Stokes).**

Impose $\partial_\mu S^\mu \geq 0$ **via a quadratic ansatz**

$$T \partial_\mu S^\mu = \frac{\Pi^2}{\zeta} - \frac{q_\mu q^\mu}{\kappa_q T} + \frac{\pi_{\mu\nu} \pi^{\mu\nu}}{2\eta_s} \geq 0$$

e.g., $T \partial_\mu S^\mu = \Pi X - q^\mu X_\mu + \pi^{\mu\nu} X_{\mu\nu}$ **will lead to equations of motion**

$$\Pi = \zeta X \ , \quad q^\mu = \kappa T \Delta^{\mu\nu} X_\nu \ , \quad \pi^{\mu\nu} = 2\eta_s X^{\langle \mu\nu \rangle}$$

—

Also follows from covariant transport using Grad's 14-moment approximation

$$f(x,p) \approx [1 + \tilde{C}_\alpha p^\alpha + C_{\alpha\beta} p^\alpha p^\beta] f_{eq}(x,p)$$

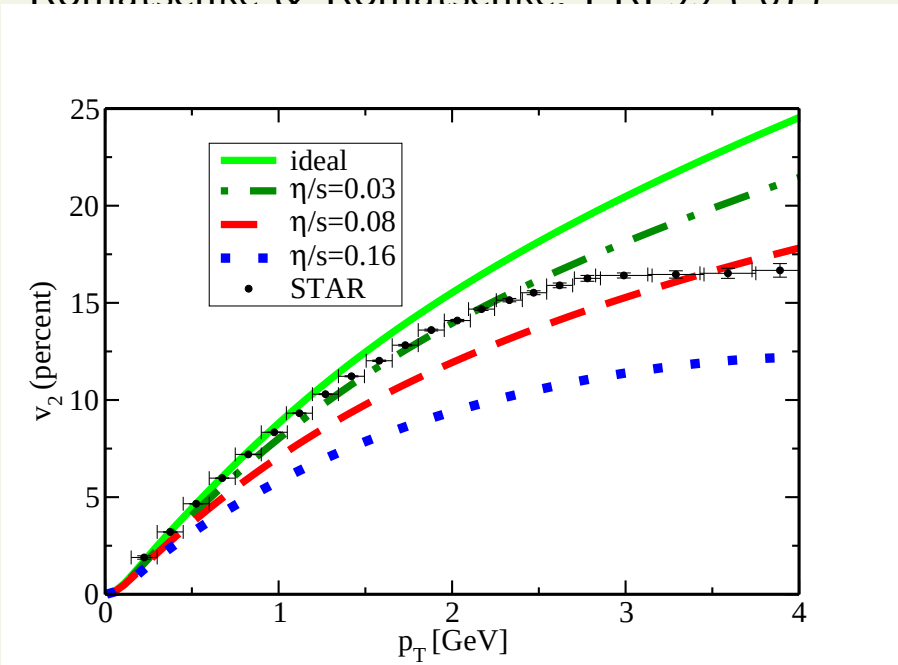
and taking the lowest three momentum moments of the Boltzmann equation.

Complete set of Israel-Stewart equations of motion

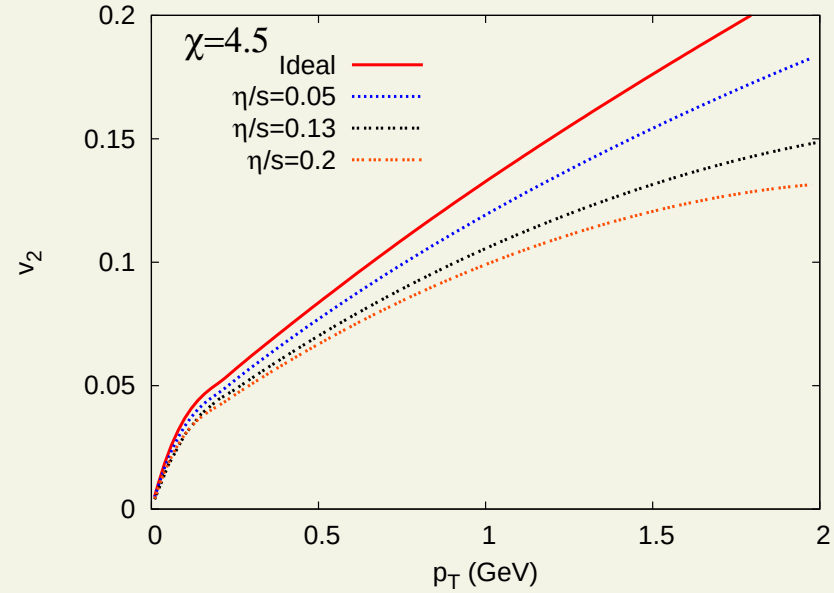
$$
\begin{aligned}
D\Pi &= -\frac{1}{\tau_\Pi}\left(\Pi + \zeta \nabla_\mu u^\mu\right) \\
&\quad -\frac{1}{2}\Pi\left(\nabla_\mu u^\mu + D\ln\frac{\beta_0}{T}\right) \\
&\quad +\frac{\alpha_0}{\beta_0}\partial_\mu q^\mu - \frac{a_0'}{\beta_0} q^\mu D u_\mu \qquad (2)\\
Dq^\mu &= -\frac{1}{\tau_q}\left[q^\mu + \kappa_q \frac{T^2 n}{\varepsilon + p}\nabla^\mu\left(\frac{\mu}{T}\right)\right] - u^\mu q_\nu D u^\nu \\
&\quad -\frac{1}{2}q^\mu\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_1}{T}\right) - \omega^{\mu\lambda} q_\lambda \\
&\quad -\frac{\alpha_0}{\beta_1}\nabla^\mu\Pi + \frac{\alpha_1}{\beta_1}(\partial_\lambda \pi^{\lambda\mu} + u^\mu \pi^{\lambda\nu}\partial_\lambda u_\nu) + \frac{a_0}{\beta_1}\Pi D u^\mu - \frac{a_1}{\beta_1}\pi^{\lambda\mu} D u_\lambda \qquad (3)\\
D\pi^{\mu\nu} &= -\frac{1}{\tau_\pi}\left(\pi^{\mu\nu} - 2\eta\nabla^{\langle\mu} u^{\nu\rangle}\right) - (\pi^{\lambda\mu}u^\nu + \pi^{\lambda\nu}u^\mu) D u_\lambda \\
&\quad -\frac{1}{2}\pi^{\mu\nu}\left(\nabla_\lambda u^\lambda + D\ln\frac{\beta_2}{T}\right) - 2\pi_\lambda{}^{\langle\mu}\omega^{\nu\rangle\lambda} \\
&\quad -\frac{\alpha_1}{\beta_2}\nabla^{\langle\mu} q^{\nu\rangle} + \frac{a_1'}{\beta_2} q^{\langle\mu} D u^{\nu\rangle}\ . \qquad (4)
\end{aligned}
$$

where $A^{\langle\mu\nu\rangle} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta}+A_{\beta\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}A^{\alpha\beta}$, $\quad \omega^{\mu\nu} \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(\partial_\beta u_\alpha - \partial_\alpha u_\beta)$

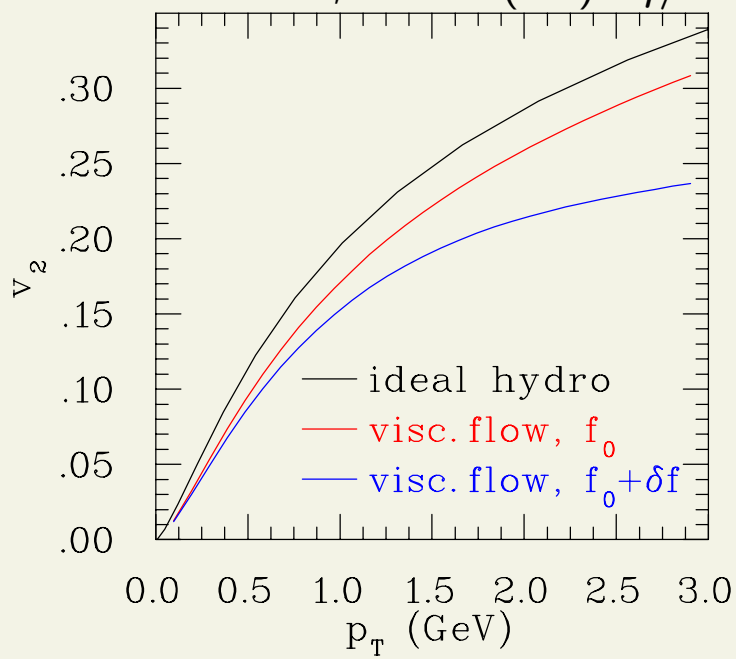
Romatschke & Romatschke, PRL 99 ('07)



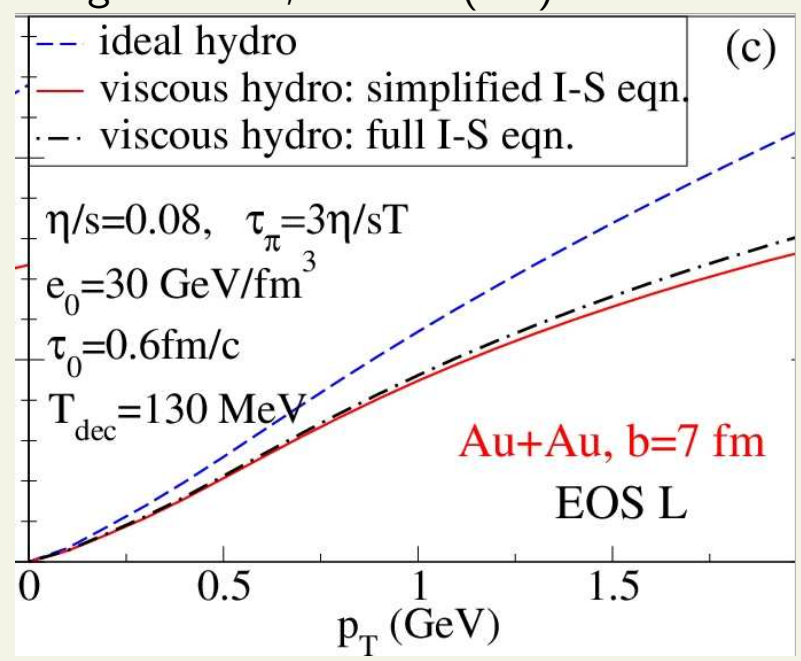
Dusling & Teaney, PRC77



Huovinen & DM, JPG35 ('08): $\eta/s \approx 1/4\pi$



Song & Heinz, PRC78 ('08)



All groups find $\sim 20-30\%$ effects for $\eta/s = 1/(4\pi)$ in Au+Au at RHIC.

Applicability of IS hydro

Israel-Stewart hydrodynamics comes from a quadratic truncation (Grad's approach) that has no small control parameter.

$\Rightarrow$ crucial to test its validity against a nonequilibrium theory

HERE: test IS hydro against $2 \to 2$ covariant transport

massless $e = 3p$ equation of state ($\zeta = 0$)

$$\eta_s \approx \frac{4T}{5\sigma_{tr}} \ , \quad \tau_\pi \approx 1.2\lambda_{tr} \equiv \frac{1.2}{n\sigma_{tr}} \qquad \sigma_{tr}: \ \text{transport cross section}$$

Two scenarios: i) $\sigma = const$ and ii) $\eta/s \approx const$ (set via $\sigma \propto time^{2/3}$).

in both cases, longitudinally expanding system $v_z = z/t$ (Bjorken flow)

[Huovinen & DM, arXiv:0808.0953]

$$T^{\mu\nu}_{LR} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p - \pi_L/2 & 0 & 0 \\ 0 & 0 & p - \pi_L/2 & 0 \\ 0 & 0 & 0 & p + \pi_L \end{pmatrix}$$

$$\dot{p} + \frac{4p}{3\tau} = -\frac{\pi_L}{3\tau} \tag{5}$$

$$\dot{\pi}_L + \frac{\pi_L}{\tau}\left(\frac{2K(\tau)}{3C} + \frac{4}{3} + \frac{\pi_L}{3p}\right) = -\frac{8p}{9\tau} \qquad (C \approx 4/5) \tag{6}$$

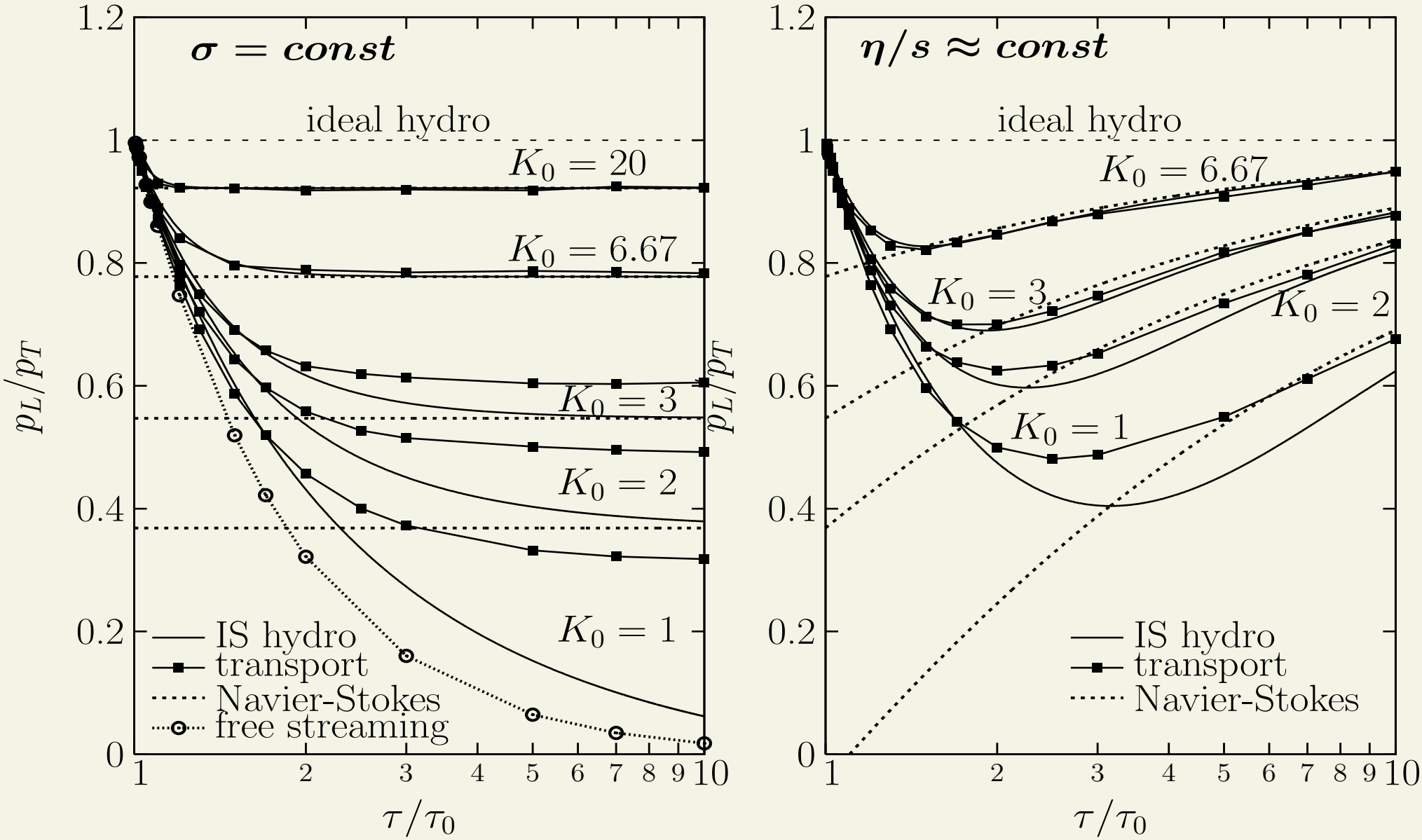
The most relevant parameter is the inverse Knudsen number

$$K(\tau) \equiv \frac{\tau}{\lambda_{tr}(\tau)} = \frac{\tau_{exp}}{\tau_{scatt}} \approx \frac{6\tau_{exp}}{5\tau_\pi} \tag{7}$$

for $\sigma = const$**:** $\lambda_{tr} = 1/n\sigma_{tr} \propto \tau \quad \Rightarrow \quad K = K_0 = const,$ $\eta/s \sim T\lambda_{tr} \sim \tau^{2/3}$

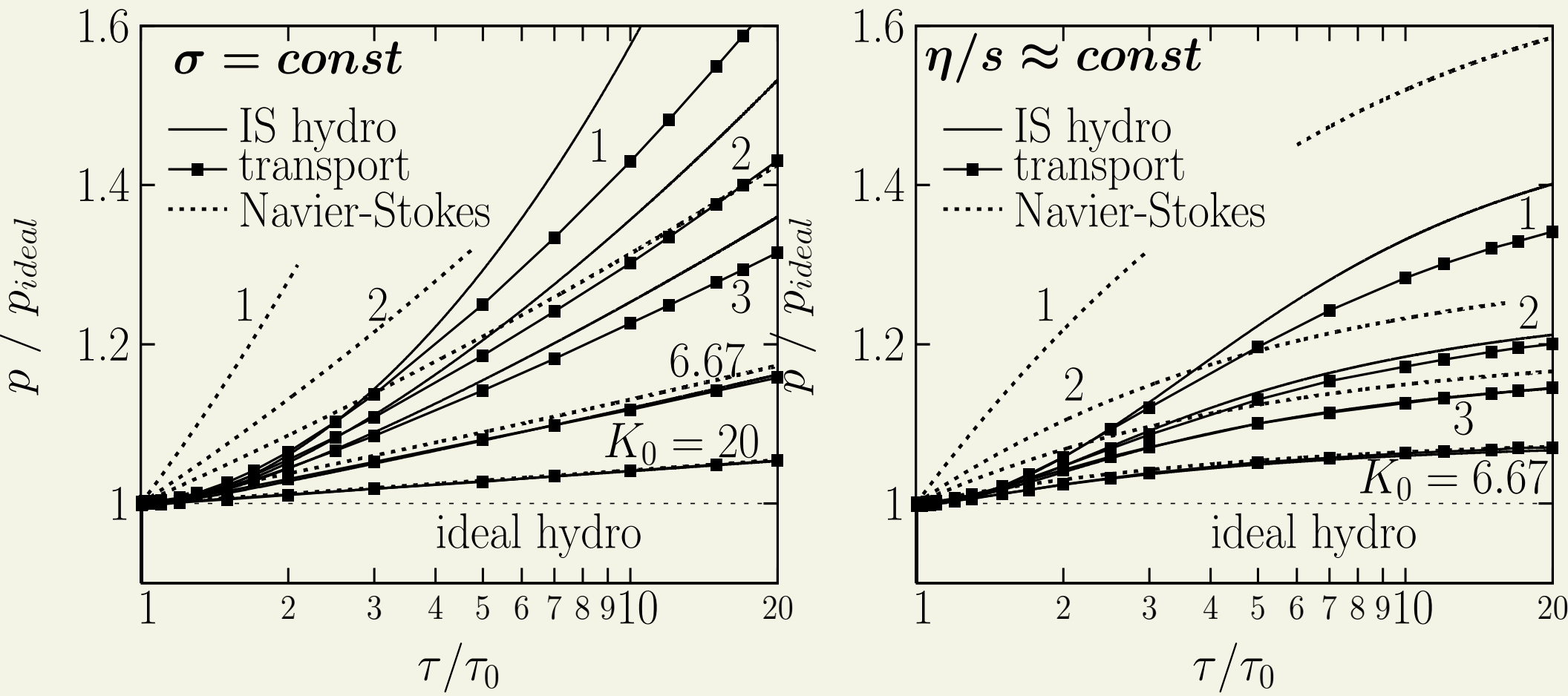
for $\eta/s \approx const$**:** $K = K_0(\tau/\tau_0)^{\sim 2/3} \propto \tau^{\sim 2/3}$

Huovinen & DM, arXiv:0808.0953 - **evolution of pressure anisotropy** T_{zz}/T_{xx}



IS hydro applicable when $K_0 \gtrsim 2-3$**, i.e.,** $\lambda_{tr} \lesssim 0.3-0.5\ \tau_0$

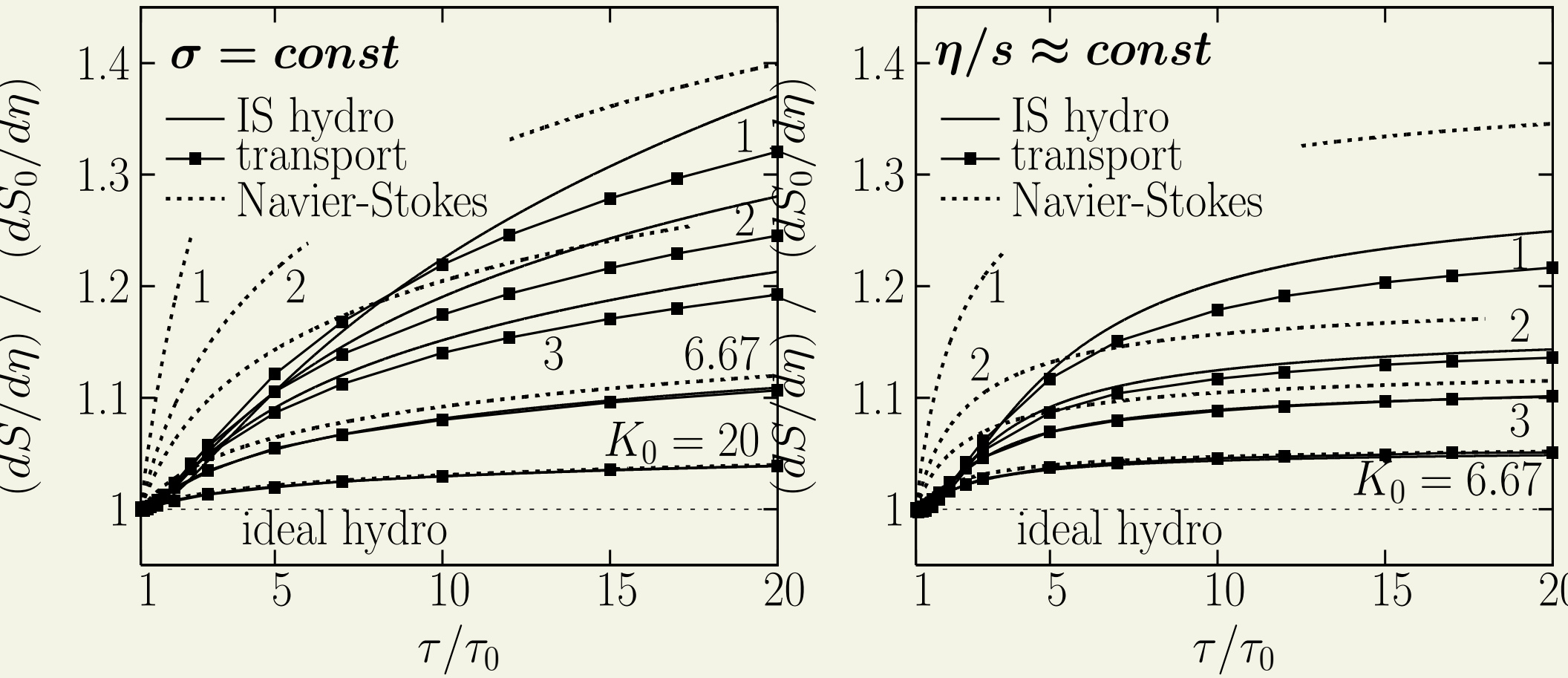
pressure evolution relative to ideal hydro



IS hydro applicable when $K_0 \gtrsim 2-3$**, i.e.,** $\lambda_{tr} \lesssim 0.3-0.5\ \tau_0$

while Navier-Stokes needs $K_0 \gtrsim 6$

entropy produced relative to initial entropy



same conclusions as before

Connection to viscosity

$$K_0 \approx \frac{T_0 \tau_0}{5} \frac{s_0}{\eta_{s,0}} \approx 12.8 \times \left(\frac{T_0}{1\ \mathrm{GeV}}\right) \left(\frac{\tau_0}{1\ \mathrm{fm}}\right) \left(\frac{1/(4\pi)}{\eta_{s_0}/s_0}\right) \tag{8}$$

For typical RHIC hydro initconds $T_0 \tau_0 \sim 1$, therefore

$$K_0 \gtrsim 2 - 3 \qquad \Rightarrow \qquad \frac{\eta}{s} \lesssim \frac{1-2}{4\pi} \tag{9}$$

I.e., shear viscosity cannot be many times more than the conjectured bound, for IS hydro to be applicable.

When IS hydro is accurate, dissipative corrections to pressure and entropy do not exceed 20% significantly.

IS hydro validity test in 2D

We solve the full Israel-Stewart equations, including vorticity terms from kinetic theory, in a 2+1D boost-invariant scenario. **Shear stress only.**

$$\dot{\pi}^{\mu\nu}+\frac{1}{\tau_\pi}\pi^{\mu\nu} = \frac{1}{\beta_2}\nabla^{\langle\mu}u^{\nu\rangle}-\frac{1}{2}\pi^{\mu\nu}D_\alpha u^\alpha-\frac{1}{2}\pi^{\mu\nu}[\ln\dot{\frac{\beta_2}{T}}]+2\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda}-(u^\mu\pi^{\nu\alpha}+u^\nu\pi^{\alpha\mu})\dot{u}_\alpha$$

Mimic a known reliable transport model:

- **massless Boltzmann particles** $\Rightarrow \epsilon = 3P$
- **only** $2 \leftrightarrow 2$ **processes, i.e. conserved particle number**
- $\eta = 4T/(5\sigma_{\rm tr})$, $\beta_2 = 3/(4p)$
- **either** σ **= const.** $= 47$ **mb (**$\sigma_{tr} = 14$ **mb)** $\leftarrow$ **the simplest in transport**
  or $\sigma \propto \tau^{2/3}$ $\Rightarrow \eta/s \approx 1/(4\pi)$

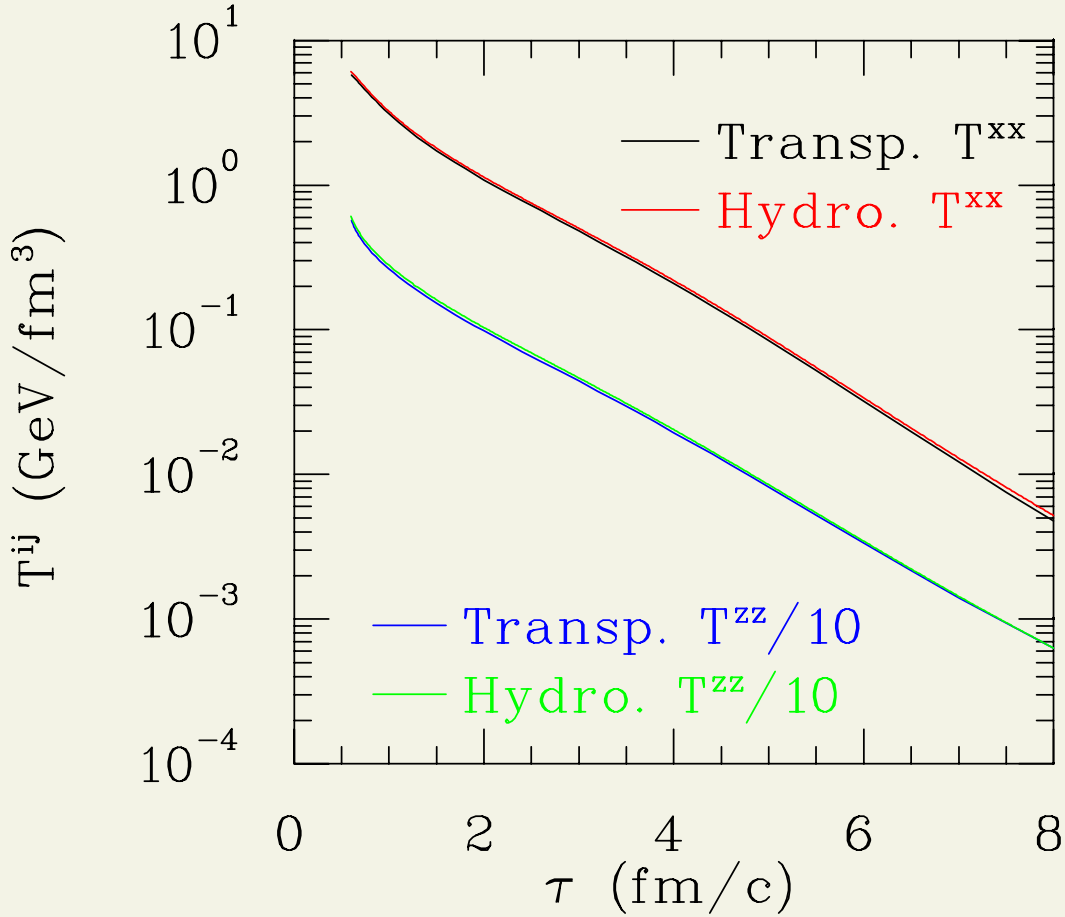
"RHIC-like" initialization:

- $\tau_0 = 0.6$ **fm/**c
- $b = 8$ **fm**
- $T_0 = 385$ **MeV and** $dN/d\eta|_{b=0} = 1000$
- **freeze-out at constant** $n = 0.365$ **fm**$^{-3}$

Pressure evolution in the core

T^{xx} **and** T^{zz} **averaged over the core of the system,** $r_\perp < 1$ **fm:**

$\boldsymbol{\eta/s \approx 1/(4\pi)}$ $\quad (\sigma \propto \tau^{2/3})$



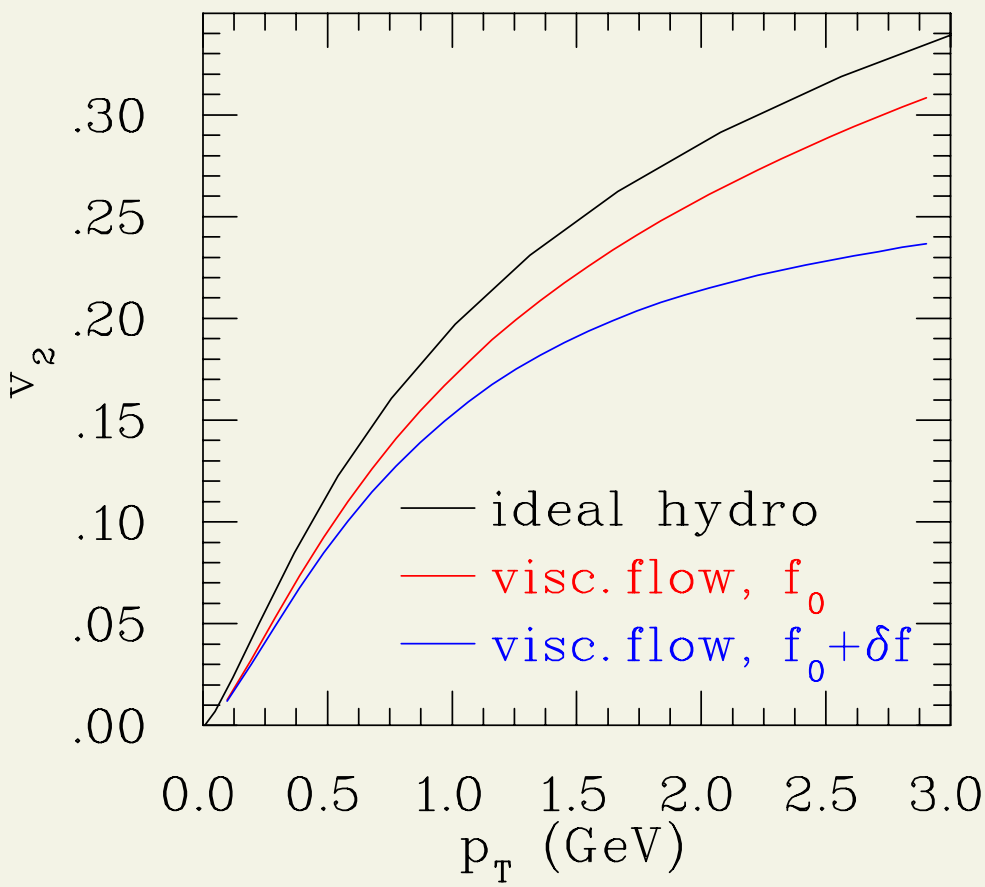
Huovinen & DM, arXiv:0806.1367

remarkable similarity!

Viscous hydro elliptic flow

TWO effects: - **dissipative corrections to hydro fields** u^μ, T, n
- **dissipative corrections in Cooper-Frye freezeout** $f \to f_0 + \delta f$

$\boldsymbol{\eta/s \approx 1/(4\pi)}$ $\quad (\sigma \propto \tau^{2/3})$



Must use Grad's quadratic correction in Cooper-Frye formula

$$E\frac{dN}{d^3p} = \int p^\mu d\sigma_\mu (f_0 + \delta f)$$

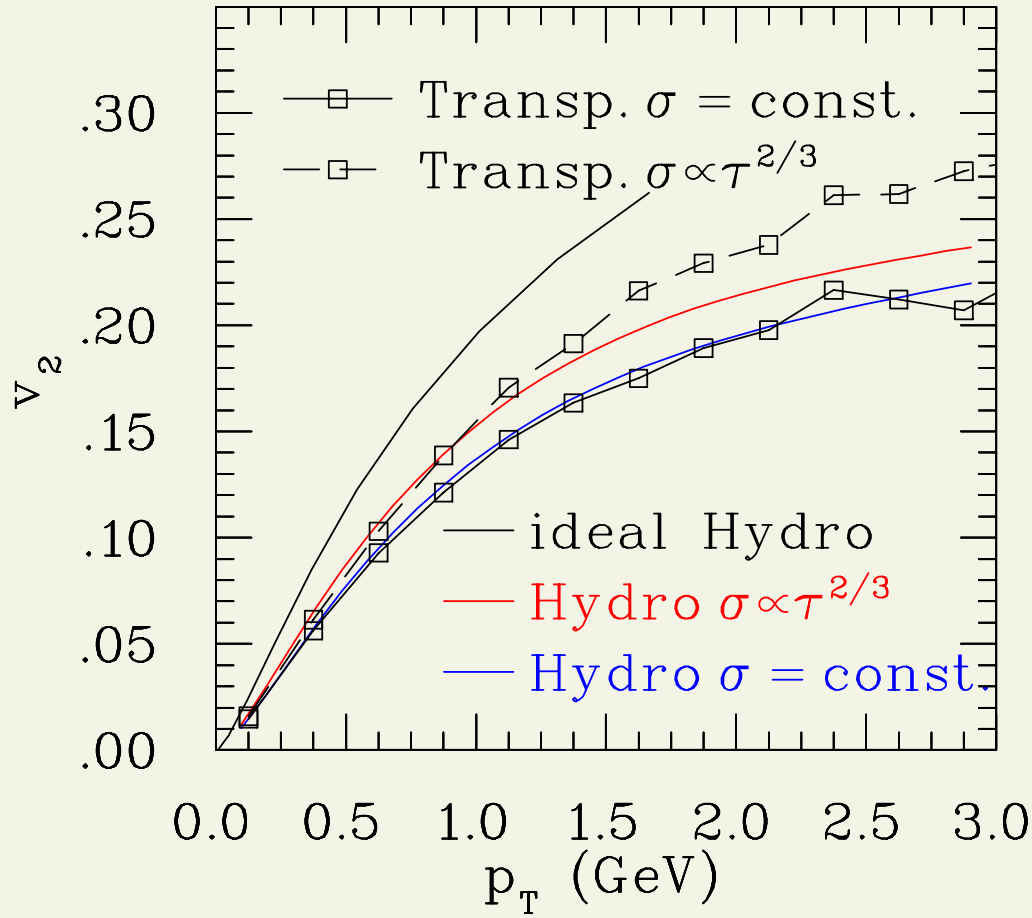
for massless $\varepsilon = 3p$**, shear only**

$$\delta f = f_0 \left[1 + \frac{p^\mu p^\nu \pi_{\mu\nu}}{8nT^3} \right]$$

calculation for $\sigma_{\rm tr} = const \sim 15mb$ **shows similar behavior**

Viscous hydro vs transport v_2

Huovinen & DM, arXiv:0806.1367



- **excellent agreement when** $\sigma =$ **const** $\sim 47mb$
- **good agreement for** $\eta/s \approx 1/(4\pi)$**, i.e.,** $\sigma \propto \tau^{2/3}$

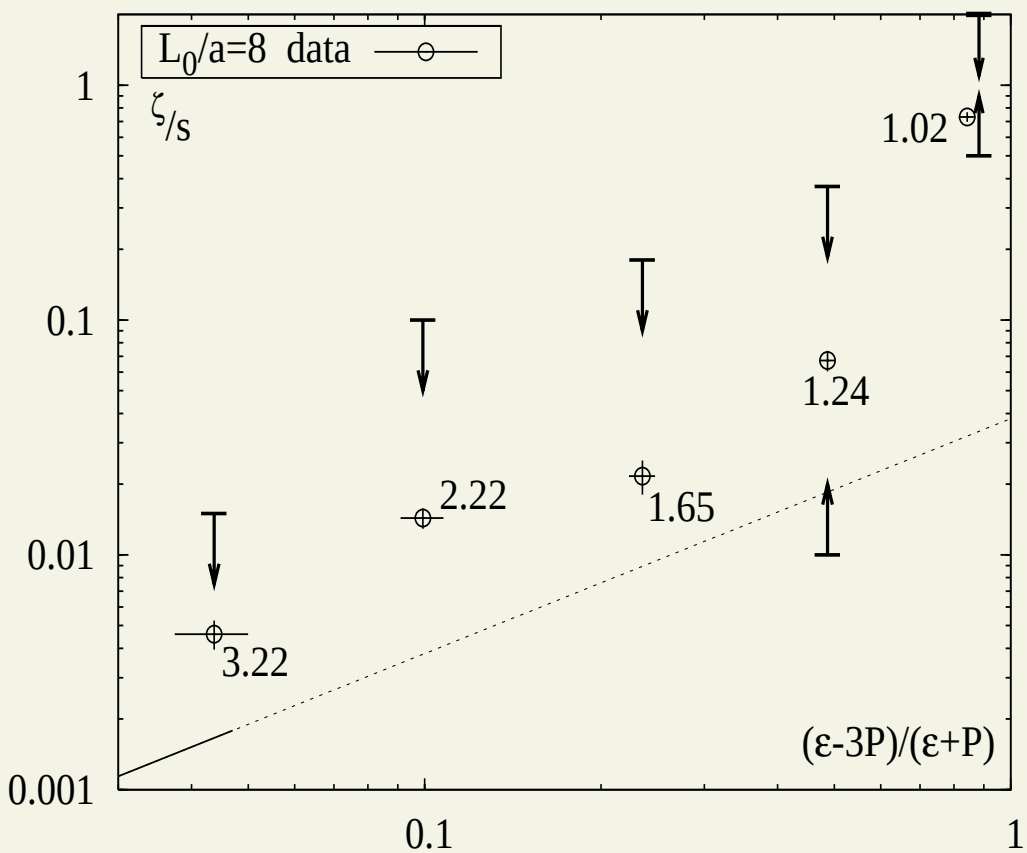
Bulk viscosity

acts against compression/dilution: $\Pi = -\zeta \vec{\nabla}\, \vec{v}$

largely unknown at $T \sim 200 - 400$ MeV relevant for RHIC

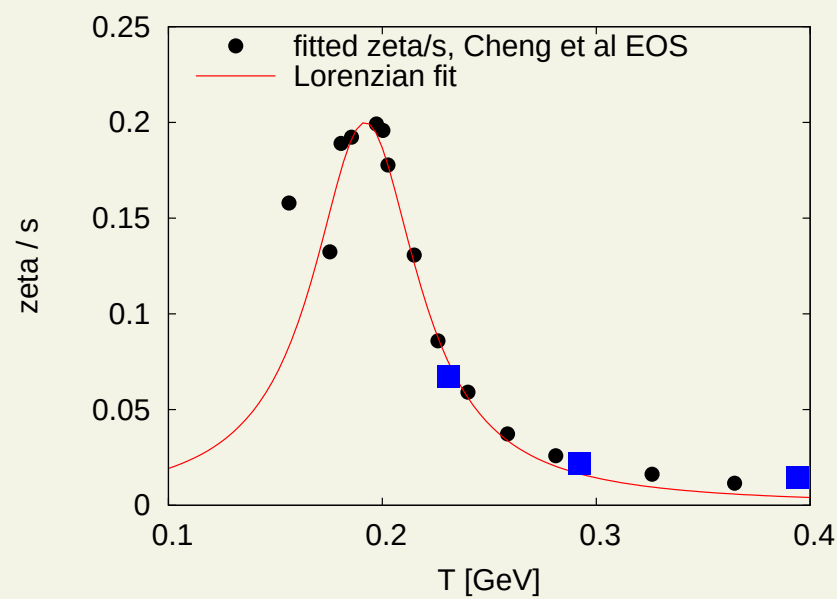
perturbatively tiny: $\zeta/s \sim 0.02\alpha_s^2$,
Arnold, Dogan, Moore, PRD74 ('06)

on lattice: very preliminary
Meyer, arXiv:0710.3717

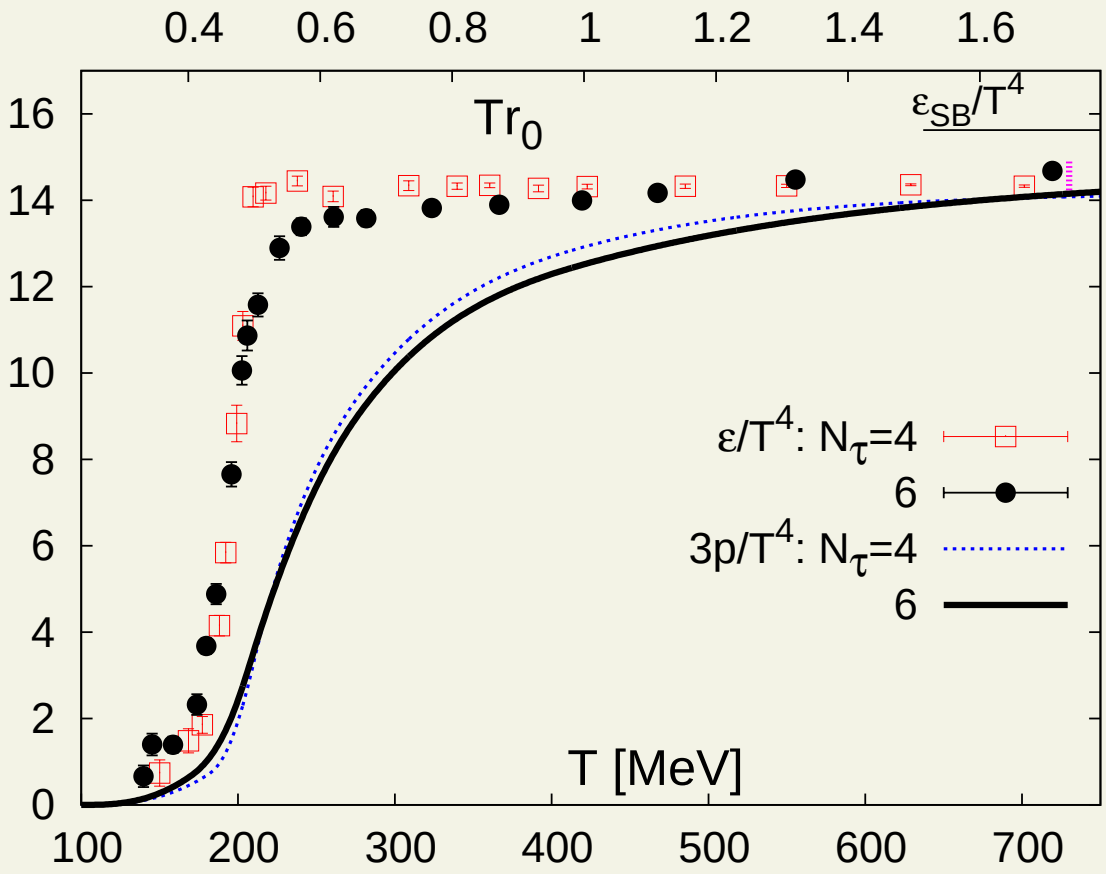


- **no quarks (gluons only)**
- **crude lattices**

implies peak near T_c

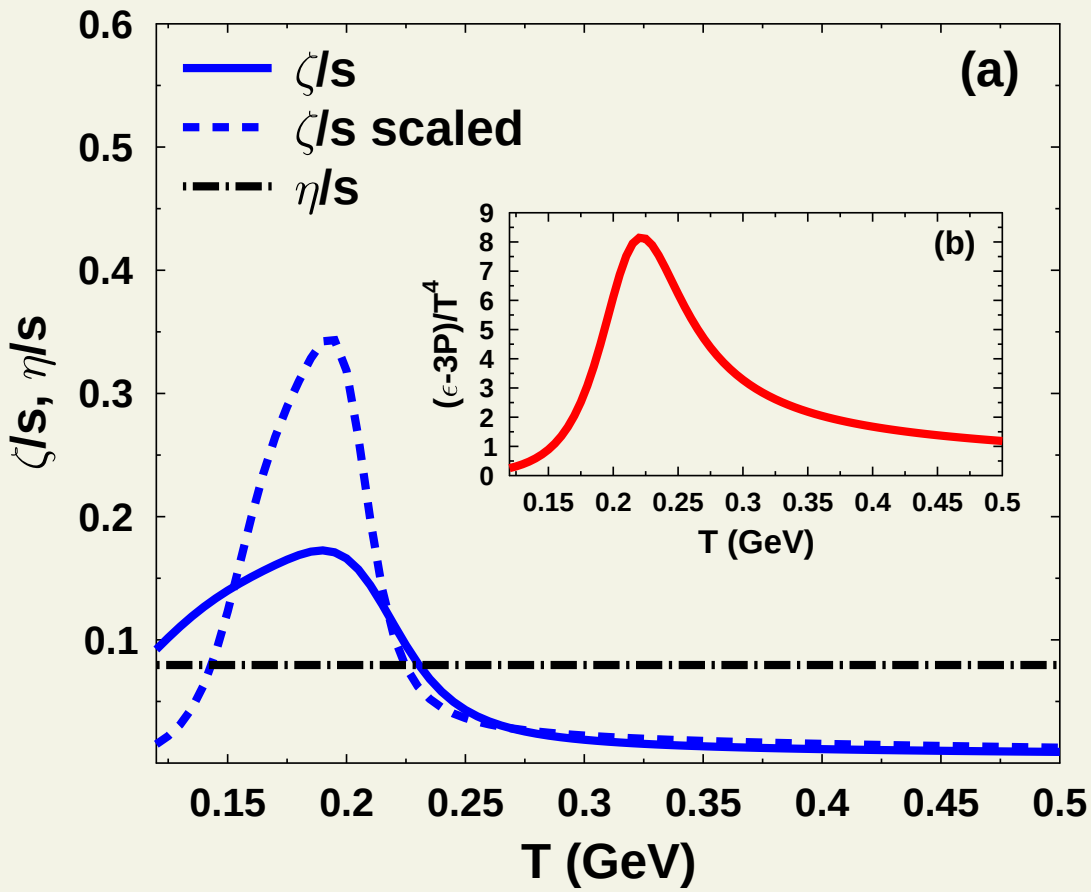


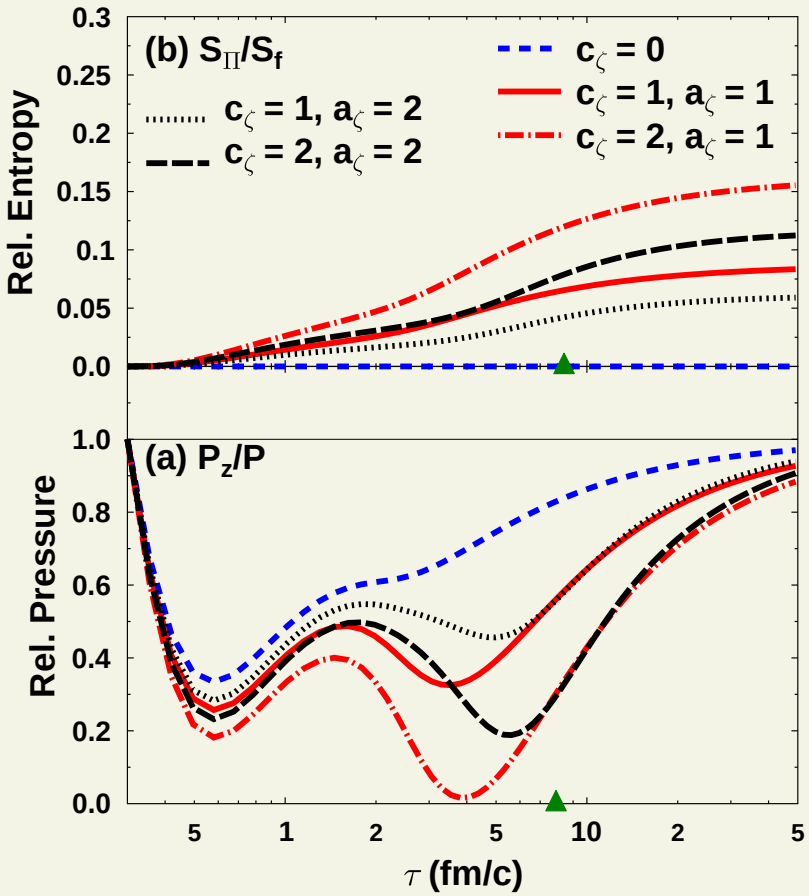
lattice QCD equation of state Cheng et al, ('07)



Fries et al, PRC78 ('08): **study in 0+1D Bjorken scenario**

lattice EOS with $\eta/s = 1/(4\pi)$, $\tau_0 = 0.3$ fm, vary width and height of $\zeta/s(T)$





finds $10-15$% entropy production from bulk viscosity possible

however, study is thermodynamically inconsistent because it ignored terms in equation of motion $D\Pi = -\frac{1}{\tau_\Pi}\left(\Pi + \zeta\nabla_\mu u^\mu\right) - \frac{1}{2}\Pi\left(\nabla_\mu u^\mu D\ln\frac{\beta_0}{T}\right)$

Entropy density in Israel-Stewart theory

$$\frac{dS}{d\eta} = \tau A_T \left[s_{eq} - \frac{1}{2T} \left(\beta_0 \Pi^2 + \frac{3}{2} \beta_2 \pi_L^2 \right) \right]$$

and only complete IS equations ensure

$$\frac{d}{d\tau}\left(\frac{dS}{d\eta}\right) = \tau A_T \left(\frac{\Pi^2}{\zeta T} + \frac{3\pi_L^2}{4\eta_s T} \right) \equiv \frac{d}{d\tau}\left(\frac{dS}{d\eta}\right)^{bulk} + \frac{d}{d\tau}\left(\frac{dS}{d\eta}\right)^{shear} \geq 0$$

The "naive" approximation violates the 2nd law in general, and also

$$\frac{d}{d\tau}\left(\frac{dS}{d\eta}\right) \neq \frac{d}{d\tau}\left(\frac{dS}{d\eta}\right)^{bulk} + \frac{d}{d\tau}\left(\frac{dS}{d\eta}\right)^{shear}$$

Bulk viscosity produces extra entropy. If too much is generated, hydro itself becomes questionable.

one benchmark: relative entropy production

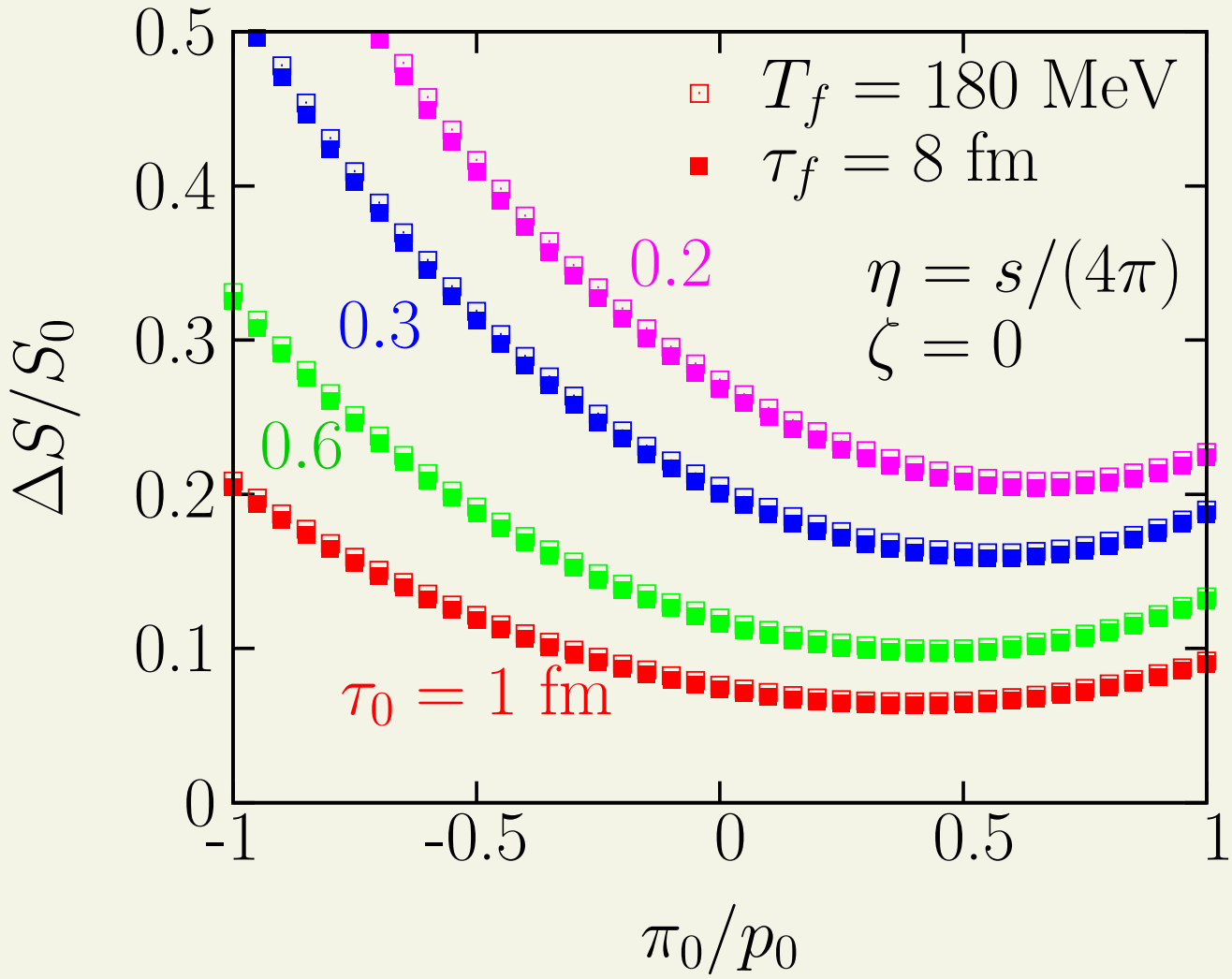
$$\frac{\Delta S^{\pi+\Pi}}{S_0} = \frac{S(T_f = 180 MeV) - S_0}{S_0} \quad \lesssim \quad 0.2 - 0.25$$

depends on initial conditions (shear stress and bulk pressure), and initial time.

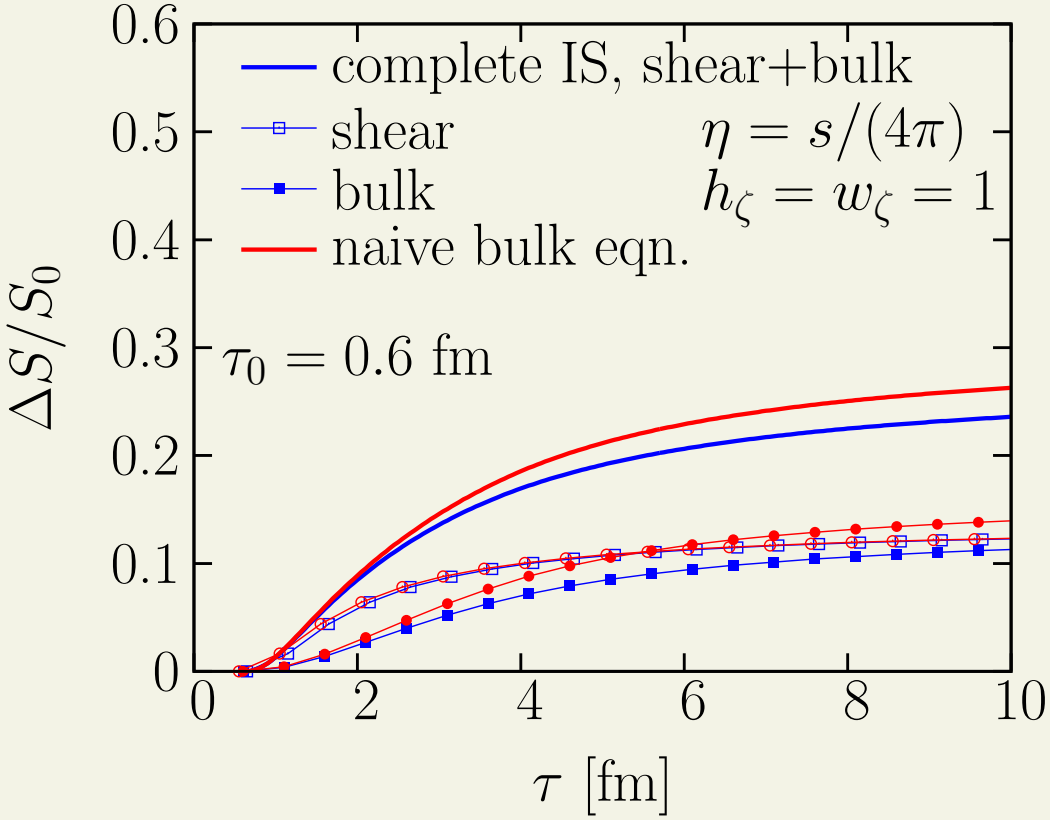
0+1D exploration for "RHIC"-like conditions:

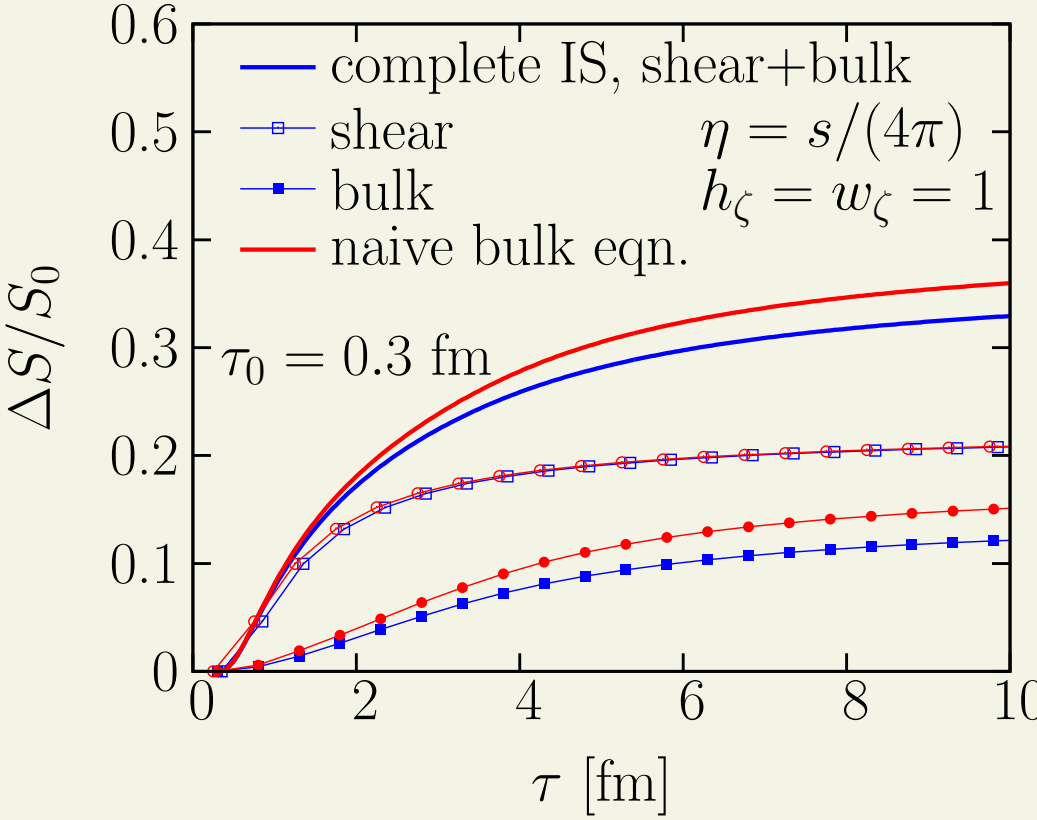
- **most recent lattice EOS, continued over to a resonance gas at low** T
- $\eta/s = 1/(4\pi)$ **fixed, vary width** w_ζ **and height** h_ζ **of** $\zeta/s(T)$**,** $\tau_\Pi = \tau_\pi = 6\eta/(sT)$
- $\varepsilon_0(\tau_0 = 0.6 fm) = 15$ **GeV/fm**^{3} **- for other** τ_0**, scale via** $s_0\tau_0 = const$ **(ideal)**
- **stop at beginning of hadron phase** $T_f = 180$ **MeV (**$\tau_f \approx 8$ **fm)**

entropy generation with shear viscosity only Huovinen & DM, ('08)



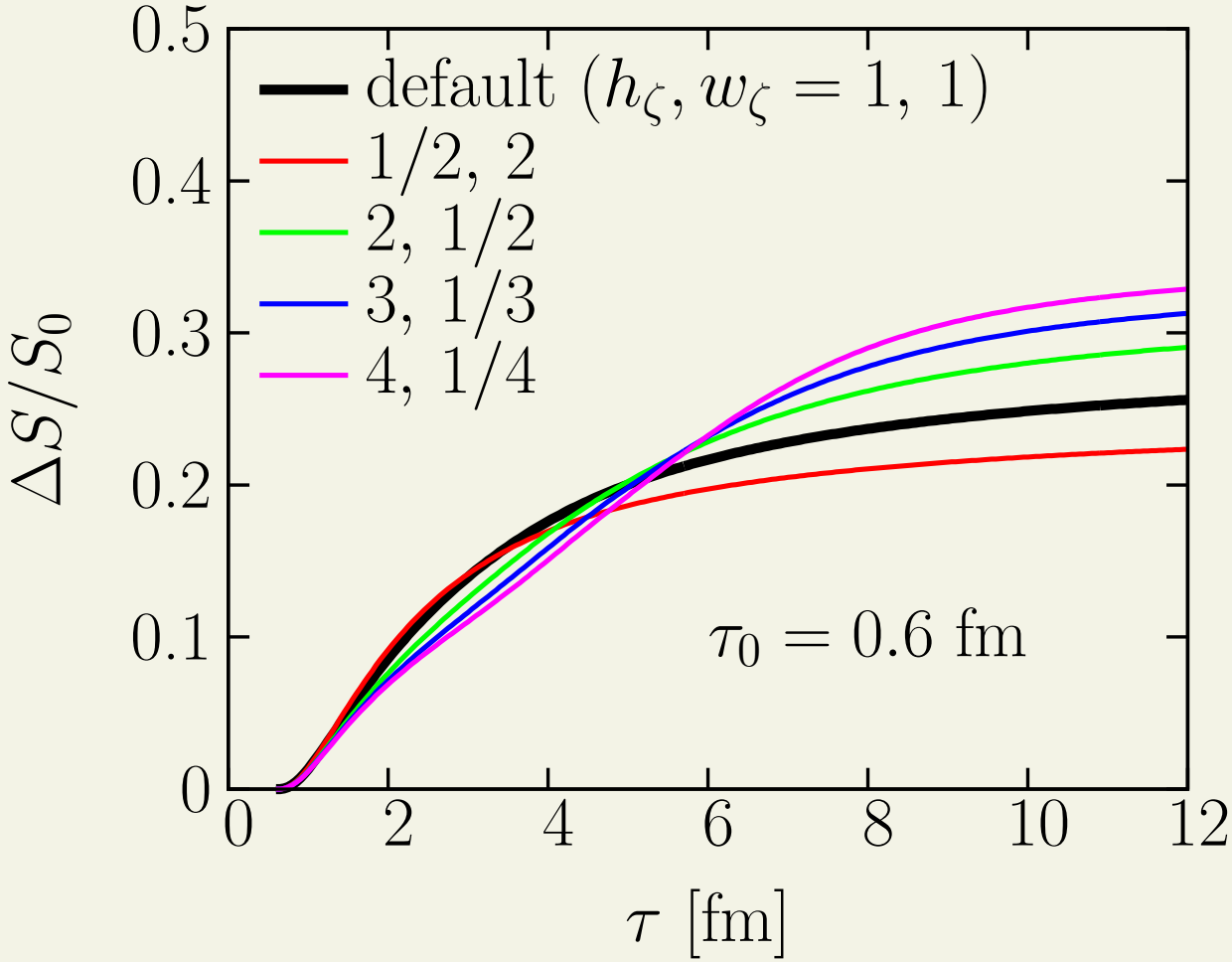
naive vs complete BULK terms in equations Huovinen & DM, ('08)



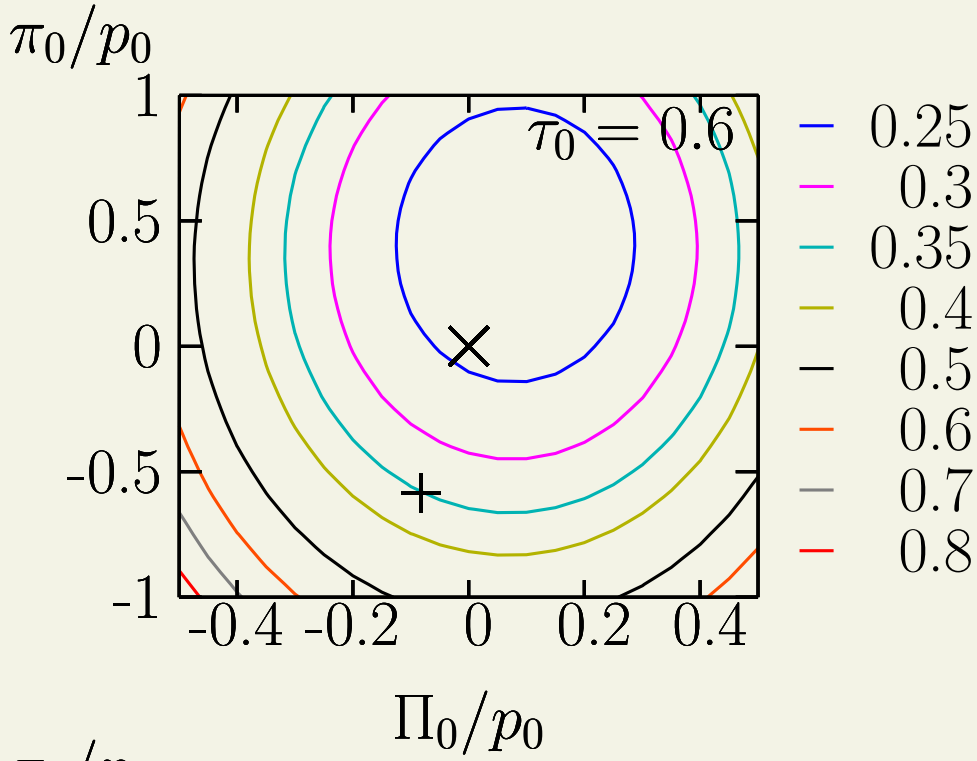


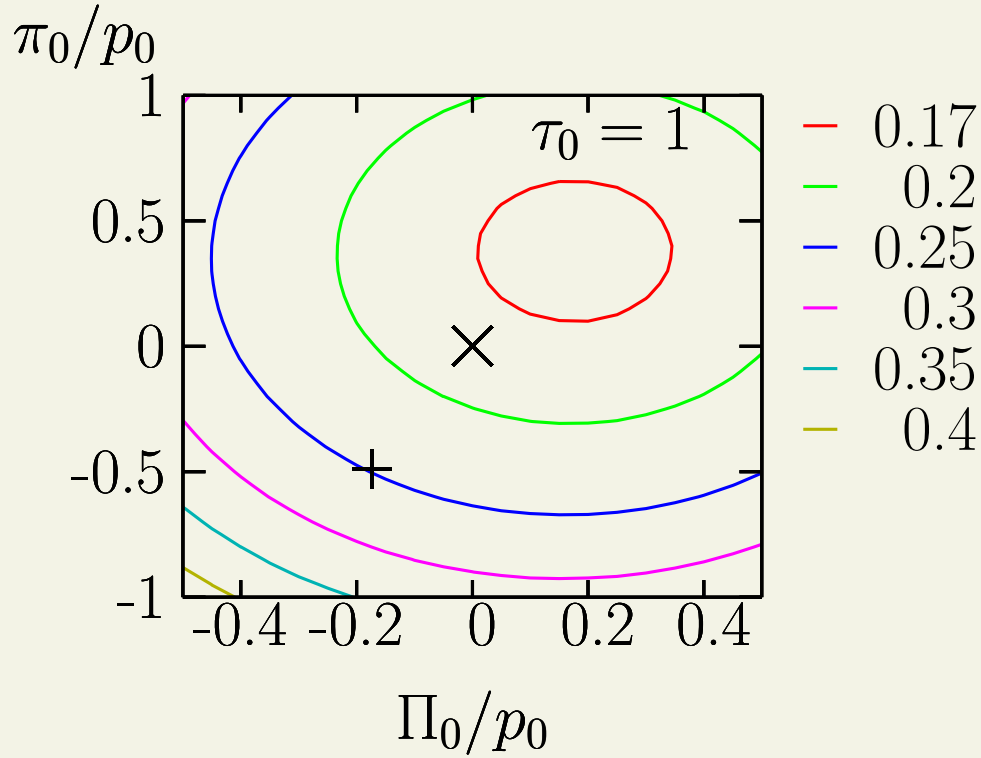
correct IS equations produce somewhat less entropy

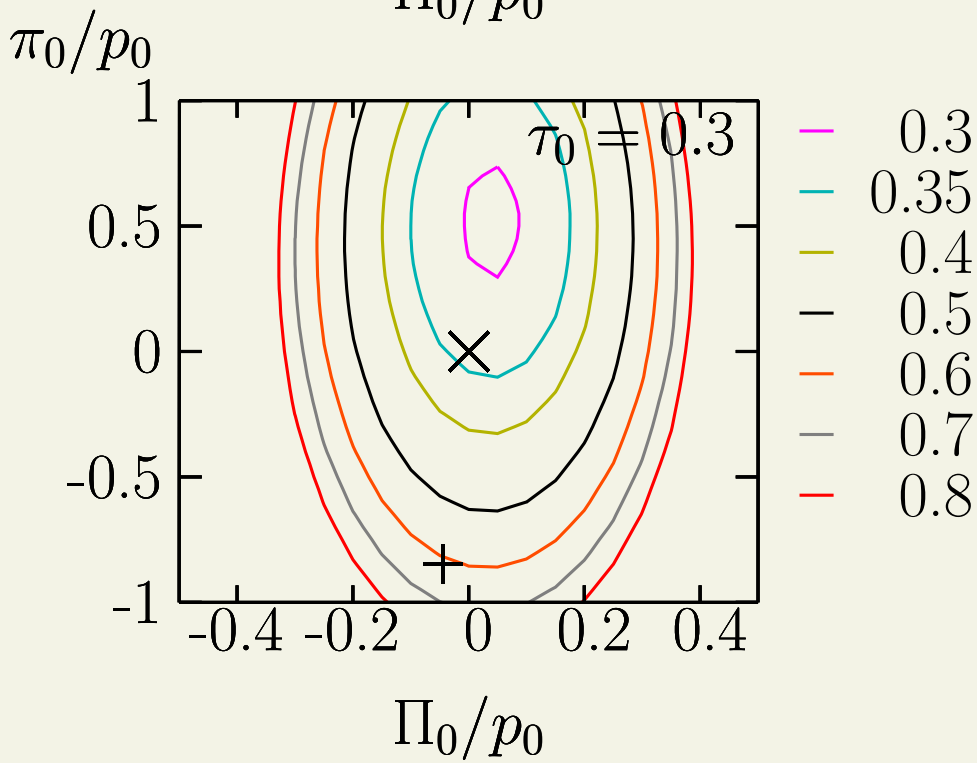
dependence on height on width of $\zeta/s(T)$ Huovinen & DM, ('08)



bulk effects significant for $h_\zeta \cdot w_\zeta \sim \mathcal{O}(1)$







requirement of modest $\leq 20 - 25\%$ entropy production severely limits initial conditions if $\tau_0 \lesssim 1$ fm

even Navier-Stokes limit is hard to accommodate

color glass $\pi_0 \sim -p_0$ is marginal even at $\tau_0 = 1$ fm

Summary

Progress so far:

- **we can solve causal viscous hydro in 2+1D**
- **IS hydro is a good approximation, if shear viscosity is not too large** $\eta/s \lesssim few/(4\pi)$

  must use complete set of Israel-Stewart equations

- $\eta/s \sim \mathcal{O}(1)/(4\pi)$ **is compatible with RHIC** $v_2(p_T)$ **data (charged hadrons),**
- **if bulk viscosity is significant (Meyer's central values), viscous hydro is applicable only for very limited initial conditions. E.g., need** $\tau_0 \gtrsim 0.5$ **fm.**

Missing ingredients:

- **establish IS hydro region of validity for realistic equation of state**
- **include bulk viscosity in 2+1D (in progress)**
- **couple hydro to hadron transport (proper freezeout)**