Second Order Hydro in QCD

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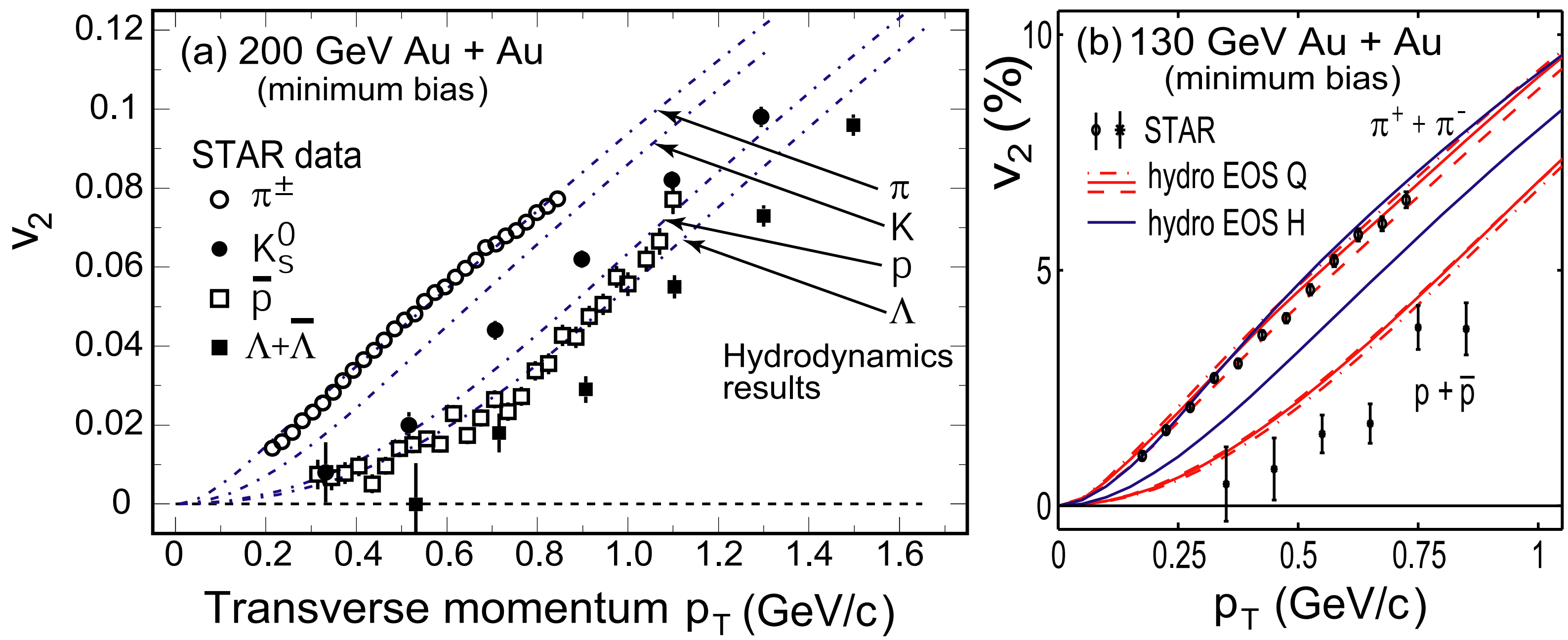
- Why do hydrodynamics in QCD?
- Why find 2'nd order coefficients and what are they?
- Kinetic theory: setup
- Kinetic theory: details
- Interesting physics along the way
- Conclusions

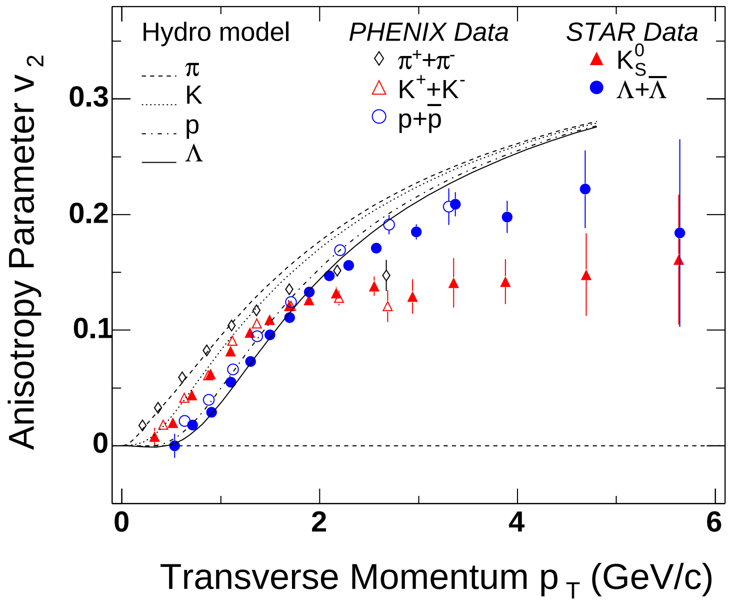
Elliptic flow is measured

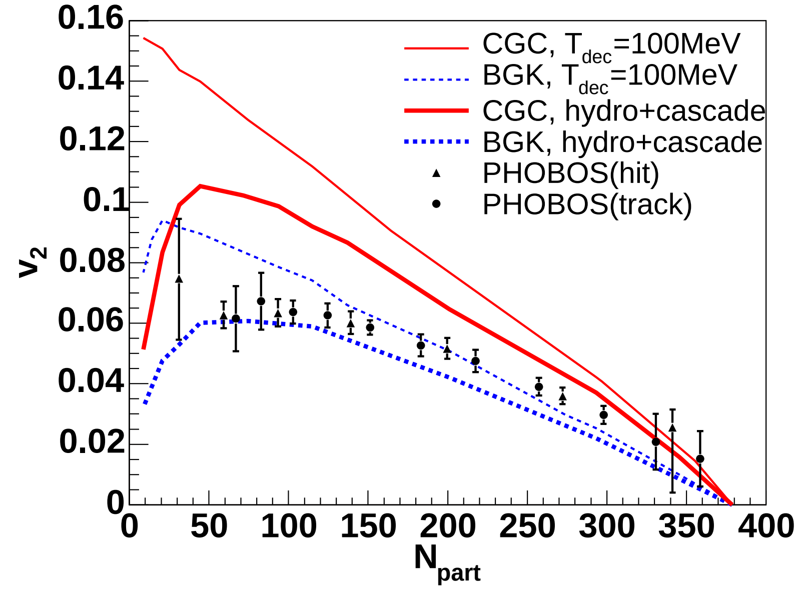


STAR experiment, minimum bias... .

We should try to understand it theoretically.

First attempt: ideal hydro





Works, DEPENDING on initial conditions.

Corrections to ideality exist, but are “small” (?)

Can we quantify that?

Ideal Hydrodynamics

Ideal hydro: stress-energy conservation

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{(4 equations, 10 unknowns)}$$

plus local equilibrium *assumption*:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\mathrm{eq}} = \epsilon u^\mu u^\nu + P(\epsilon)\Delta^{\mu\nu}, \\ u^\mu u_\mu &= -1, \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \end{aligned}$$

depends on 4 parameters (ϵ, 3 comp of u^μ): closed.

Ideal hydro works well: corrections *eg, viscosity* small
Claim: "Most Perfect Liquid" exotic behavior. **Quantify!**

Nonideal Hydro

Assume that ideal hydro is "good starting point," look for small systematic corrections.

Near equilibrium iff $t_{\rm therm} \ll t_{\rm vary}, l_{\rm vary}/v$ (so ∂ small)

Allows expansion of corrections in gradients:

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{\rm eq} + \Pi^{\mu\nu}[\partial, \epsilon, u] \\ \Pi^{\mu\nu} &= \mathcal{O}(\partial\mu, \partial\epsilon) + \mathcal{O}(\partial^2\mu, (\partial\mu)^2, \ldots) + \mathcal{O}(\partial^3 \ldots) \end{aligned}$$

For Conformal theory $T^\mu_\mu = 0 = \Pi^\mu_\mu$, 1-order term unique:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu}\,, \quad \sigma^{\mu\nu} = \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta}\partial\cdot u\right)$$

Coefficient η is shear viscosity.

Viscous hydro

So why not consider (Navier-Stokes)

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} \quad ?$$

Because in **relativisitc** setting, it is

- Acausal: shear viscosity is transverse momentum diffusion. Diffusion $\partial_t P_\perp \sim \nabla^2 P_\perp$ has instantaneous prop. speed. Müller 1967, Israel+Stewart 1976
- Unstable: $v > c$ prop + non-uniform flow velocity $\to$ propagate from future into past, exponentially growing solutions. Hiscock 1983

Problem only on short length scales where $\eta|\sigma| \sim P$. But numerics must treat these scales (or "numerical viscosity" which exceeds η is present)

Israel-Stewart approach

Add one second order term:

$$\Pi^{\mu\nu} = -\eta\sigma^{\mu\nu} + \eta\tau_\Pi\, u^\alpha\partial_\alpha\sigma^{\mu\nu}$$

Make (1'st order accurate) $\eta\sigma \longrightarrow -\Pi$ in order-2 term:

$$\tau_\Pi\, u^\alpha\partial_\alpha\Pi^{\mu\nu} \equiv \tau_\Pi\, \dot{\Pi}^{\mu\nu} = -\eta\sigma^{\mu\nu} - \Pi^{\mu\nu}$$

Relaxation eq driving $\Pi^{\mu\nu}$ towards $-\eta\sigma^{\mu\nu}$.

Momentum diff. no longer instantaneous.

Causality, stability are restored (depending on τ_Π)

But why only one 2'nd order term???

Second order hydrodynamics

It is more consistent to include all possible 2'nd order terms.
Assume *conformality* and *vanishing chem. potentials*:
5 possible terms Baier *et al*, [arXiv:0712.2451]

$$\begin{aligned}
\Pi^{\mu\nu}_{2\text{ ord.}} = \; & \eta\tau_\Pi \left[u^\alpha\partial_\alpha\sigma^{\mu\nu} + \frac{1}{3}\sigma^{\mu\nu}\partial_\alpha u^\alpha\right] + \lambda_1\left[\sigma^\mu_\alpha\sigma^{\nu\alpha} - (\text{trace})\right] \\
& +\lambda_2\left[\frac{1}{2}(\sigma^\mu_\alpha\Omega^{\nu\alpha} + \sigma^\nu_\alpha\Omega^{\mu\alpha}) - (\text{trace})\right] \\
& +\lambda_3\left[\Omega^\mu{}_\alpha\Omega^{\nu\alpha} - (\text{trace})\right] + \kappa\left(R^{\mu\nu} - \ldots\right) \; , \\
\Omega_{\mu\nu} \equiv \; & \frac{1}{2}\Delta_{\mu\alpha}\Delta_{\nu\beta}(\partial^\alpha u^\beta - \partial^\beta u^\alpha) \;\; [\text{vorticity}] \, .
\end{aligned}$$

Now, besides η, we have 5 more unknown coefficients.

Second order: philosophy

(nonideal) hydro only consistent if η makes small corr.

These τ_Π, $\lambda_{1,2,3}$ make smaller corrections. κ irrelevant.

Make reasonable estimate for τ_Π, λ_{123}, test sensitivity

Guy argues: Ratios should be robust

- Forget $\frac{\eta}{s}$. Think of $\frac{\eta}{P+\epsilon} = t_\eta$ a *timescale*.
- Next order: $\frac{\lambda_1}{P+\epsilon} = t_\lambda^2$, $\frac{\eta\tau_\Pi}{P_\epsilon} = t_\pi^2$
- All are thermalization times. One expects $t_\lambda \sim t_\eta \sim t_\pi$.

Determine ratios where you can, use as priors in fit

Two toy models of QCD

To date, coeff's computed in *toy model* for QCD:
$\mathcal{N}{=}4$ SYM theory at $N_{\rm c}, g^2 N_{\rm c} \to \infty$
(conformal, vast number DOF, many scalars, infinite coupling,...) Baier et al [arXiv:0712.2451], Tata group, [arXiv:0712.2456]

I know another *toy model* for QCD:
Weakly coupled $N_{\rm c} = 3$, $N_{\rm f} = 0, \ldots 6$ QCD in pert. theory!
(asymptotically free, mass gap, right number DOF, finite coupling...)

Leading order calculation: theory conformal, same coeff's
Toolkit for calculation: kinetic theory (valid at leading order)

Kinetic theory

Weak coupling: IR-safe corr. funcs nearly Gaussian. Adequate description in terms of 2-point function.

Value of 2-pt function has interpretation as particle number: $\phi^\dagger \phi$ is $\frac{1}{2} + \hat{N}$ number operator of free thy.

Leading-order: free propagation. Scatterings "rare".

Allows extra approximation: $\Delta x \sim 1/p \sim 1/T$ small compared to free path $\lambda \sim 1/g^2 T$. Propagation classical, $[x, p] \simeq 0$ "classical phase space" behavior.

Kinetic theory

State, all measurables described by particle distrib. $f_a(x,p)$:

$$T^{\mu\nu}(x) = \sum_a \int_p 2p^\mu p^\nu f_a(x,p) \,, \qquad \int_p \equiv \int \frac{d^3p}{(2\pi)^3 2p^0}$$

(Assumes weak coupling, slow x^μ dependence, little else)

Dynamics: Boltzmann equation (Schwinger-Dyson eq):

$$2P^\mu \partial_{\mu_x} f(x,p) \;\; = \;\; -\mathcal{C}[p, f(x,q)]$$

LHS: particle propagation. $p^0 \equiv \sqrt{\mathbf{p}^2} \equiv p$

RHS: scattering (Im self-energy). Local in x but not p.

Theory dependence all contained in detailed form of $\mathcal{C}[f]$.

In our case, described in detail in AMY5: `hep-ph:0209353`

Two gradient expansions

Hydrodynamics description relies on

$$(t_{\rm therm}, vt_{\rm therm}) \ll (t_{\rm vary}, l_{\rm vary})$$

slow variation in time,space. Expandable order-by-order.

Kinetic theory description relies on

$$(t_{\rm deBroglie} \sim T^{-1}, \lambda_{\rm deBroglie}) \ll (\Gamma^{-1}, \lambda_{\rm mfp})$$

where $\Gamma^{-1} \leq t_{\rm therm}$ is inverse scatt rate.
We don't know how to do this expansion order-by-order!
g^2 corrections lie outside kinetic description.

Expansion in (hydro) gradients

$$f(x,p) = f_0(\beta(x), u(x), p) + f_1(\partial, \beta, u, p) + f_2(\partial^2, \beta, u, p)$$

Subscript counts order in derivatives. β, u and $\epsilon, \vec{P}$ dual

LHS of Boltzmann has 1 deriv: RHS has 0.

$$\mathcal{O}\ 0: \quad \mathcal{C}[p, f_0(x,q)] = 0 \quad \rightarrow \quad f_{0,a} = \frac{1}{\exp(-\beta u^\mu P_\mu) \pm 1}$$

$$\mathcal{O}\ 1: \quad 2P^\mu \partial_{\mu_x} f_0(\beta(x), u(x), p) = -\mathcal{C}_1[p, f(q)]$$

where $\mathcal{C}_1$ is $\mathcal{C}$ expanded to lin. order in f_1.

$$\mathcal{O}\ 2: \quad 2P^\mu \partial_{\mu_x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, fq]$$

with $\mathcal{C}_{11}$ 2 order in f_1, $\mathcal{C}_2$ lin. order in f_2

First order in expansion

Organize it as

$$f_1(q) = -\mathcal{C}^{-1}_{1,qp} 2P^\mu \partial_\mu f_0(-\beta u^\nu P_\nu)$$

Gradients of free-theory distribution act as source for f_1.

$$2P^\mu \partial_\mu f_0 = -f_0' \beta P^\mu P^\nu (\partial_\mu u_\nu + u_\mu \partial_\nu \beta)$$

Organize source in spherical harmonics. $\ell = 0, 1$ determine u, β:

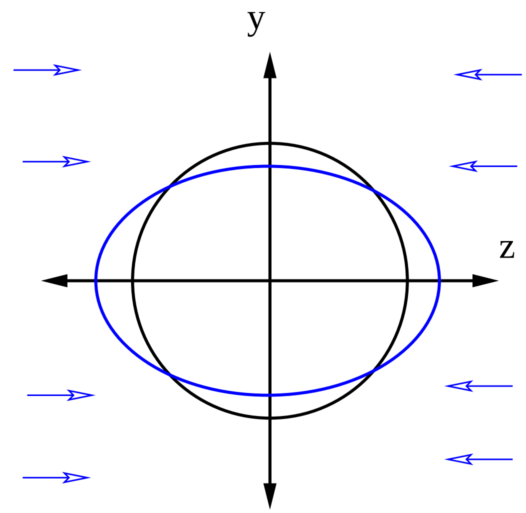
$$\partial_t \beta = \frac{\beta}{3} \partial_i u_i \qquad \text{and} \qquad \partial_t u_i = \frac{1}{\beta} \partial_i \beta$$

Remaining term is nontrivial:

$$f_1 = \mathcal{C}_1^{-1} (p_i p_j - p^2 \delta_{ij}/3) \beta \sigma_{ij} f_0'$$

Physics so far: viscosity

Consider system under compression: momentum distribution becomes prolate spheroidal (limited by scattering)



Viscosity measures $T_{\mu\nu}$ of this distortion, so $\eta/P = 4\eta/(P+\epsilon)$ measures extent of prolateness.

Prolateness can differ at different $|p|$;
$\chi(p)$ tells how this varies, η gives some average.

Second order Boltzmann Equation

$$2P^{\mu}\partial_{\mu_x} f_1 = -\mathcal{C}_{11}[p, f(q)] - \mathcal{C}_2[p, fq]$$

Organize it as

$$f_2 = -\mathcal{C}_1^{-1}\Big(2P^{\mu}\partial_{\mu} f_1 + \mathcal{C}_{11}[f_1]\Big)$$

Term on right acts like a source for 2'nd order departure f_2.
Two pieces: effects inhomogeneity on 1'st order departure, nonlinearity of collision operator in departure from equilib.

Determining Π^{ij} only requires $\ell = 2$ moment of f_2, which simplifies calculation: only need $\ell = 2$ of RHS.

Term $2P^\mu \partial_\mu f_1$

Inhomogeneous flow when f already skewed. Consider

$$2P^\mu \partial_\mu \sigma_{\alpha\beta}(P^\alpha P^\beta - g^{\alpha\beta} p^2/3)\beta^3 \chi(-\beta u^\gamma P_\gamma)$$

Two types of terms: ∂_μ acts on $\sigma_{\alpha\beta}\beta^3$, or on χ.

First term:

$$P^\mu P^\nu P^\alpha \left(\partial_\mu \sigma_{nu\alpha} + 3\sigma_{\nu\alpha}\partial_\mu \ln\beta\right)\beta^3\chi$$

Only contributions to Π^{ij} when 2 P's space, one time.

$$E p_i p_j \left(\partial_0 \sigma_{ij} + 2\partial_i \sigma_{0j} + 3\sigma_{ij}\partial_0 \ln\beta\right)$$

Contributes to τ_Π, λ_2, λ_1. Second term: contributes to λ_1.

What we get so far

One contribution to τ_Π, λ_2, and λ_1.
Automatically in ratio $1 : -2 : -1$. Therefore

$$\lambda_2 = -2\eta\tau_\Pi$$

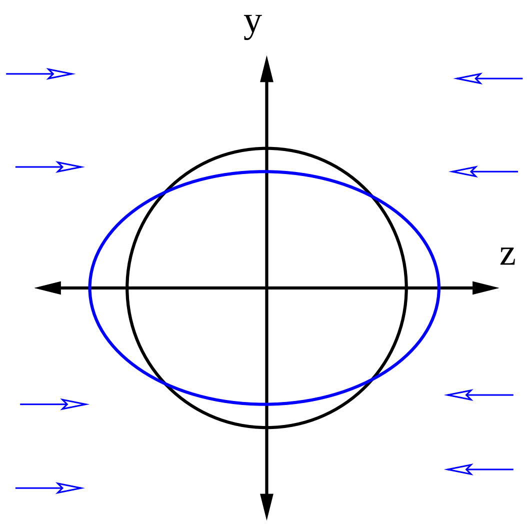
Extra independent (positive) contribution to λ_1.

Detailed values depend on functional form of $\chi(\beta E)$.
Specific *Ansatz* (Grad 14-moment) gives specific values:

$$\frac{\eta\tau_\Pi}{\epsilon + P} = \frac{6\zeta(4)\zeta(6)}{\zeta^2(5)} \left(\frac{\eta}{\epsilon + P}\right)^2 , \quad \frac{\lambda_1}{\eta\tau_\Pi} = -\frac{1}{3} \, .$$

But we *solve* for $\chi(\beta E)$–slightly different value

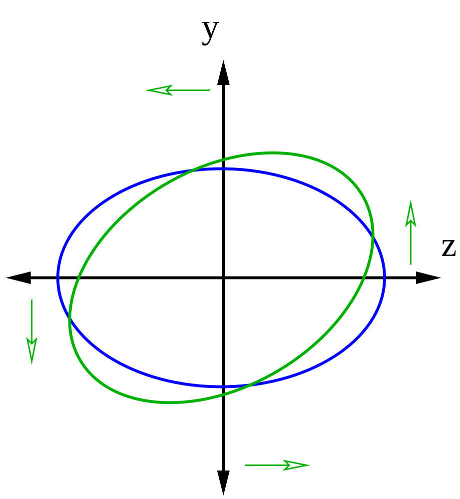
What do these coefficients mean?



Contraction $\longrightarrow$ prolateness.

τ_Π: how long prolateness lasts after contraction ends.

λ_1: induced prolateness depends on how prolate it already is!



(Vorticity) rotation

Vorticity means rotation.

λ_2: how much prolateness gets rotated in a rotating system.

Depends on how long prolateness lives.

Hence relation $\lambda_2 = -2\eta\tau_\Pi$.

Hard part: $\mathcal{C}_{11}$

Nonlinear term in collision operator:

"extra" particles scattering from other "extra" particles

Theory dependent. Consider $2 \leftrightarrow 2$ scattering (present in most theories):

$$\mathcal{C}[p, f[q]] = \int_{kp'k'} (2\pi)^4 \delta^4(P+K-P'-K') |\mathcal{M}^2| \times \\ \Big(f(p) f(k) [1 \pm f(p')][1 \pm f(k')] - f(p') f(k') [1 \pm f(p)][1 \pm f(k)] \Big)$$

First order expansion: define $\bar{f}_1 = f_0(1 \pm f_0) f_1$.

$$\Big(f(p) f(k) [1 \pm f(p')][1 \pm f(k')] - f(p') f(k') [1 \pm f(p)][1 \pm f(k)] \Big) \\ = \quad 0 + f_0(p) f_0(k) [1 \pm f_0(p')][1 \pm f_0(k')] \Big(\bar{f}_1(p) + \bar{f}_1(k) - \bar{f}_1(p') - \bar{f}_1(k') \Big)$$

That's what we needed in defining $\mathcal{C}_1$. Used twice already!

Next order: $f_0(p)f_0(k)[1\pm f_0(p')][1\pm f_0(k')]$ times

$$\bar{f}_1(p)\bar{f}_1(k)f_0(p)f_0(k)(e^{\frac{p+k}{T}}-1)+\bar{f}_1(p')\bar{f}_1(k')f_0(p')f_0(k')(1-e^{\frac{p+k}{T}})$$
$$+\Big[\bar{f}_1(p)\bar{f}_1(p')f_0(p)f_0(p')\left(e^{\frac{p}{T}}-e^{\frac{p'}{T}}\right)+(p'\to k')$$
$$+(p\to k)+(p,p'\to k,k')\Big]$$

Note that $\bar{f}_1(p)\propto\sigma_{ij}(p_ip_j-\delta_{ij}p^2/3)$.
In evaluating $\langle S_{ij}|\mathcal{C}_{11}[f_1]\rangle$ we meet angular integrations:

$$\text{defining } p_{\langle i}q_{j\rangle}=\frac{3p_iq_j+3q_ip_j-2p\cdot q\delta_{ij}}{6}\,,x_{pq}\equiv p\cdot q\,,$$
$$\sigma_{lm}\sigma_{rs}\int d\Omega_{\rm global}p_{\langle i}p_{j\rangle}q_{\langle l}q_{m\rangle}r_{\langle r}r_{s\rangle}$$
$$=\quad\frac{4}{105}\left(\sigma_{il}\sigma_{jl}-\frac{\delta_{ij}}{3}\sigma_{lm}\sigma_{lm}\right)$$
$$\times\Big(3x_{pq}x_{pr}x_{qr}-x_{pp}x_{qr}^2-x_{qq}x_{pr}^2-x_{rr}x_{pq}^2+2x_{pp}x_{qq}x_{rr}/3\Big)$$

Using these, one can bludgeon $\mathcal{C}_{11}$ term to death. Contributes only to λ_1.

Subtlety!

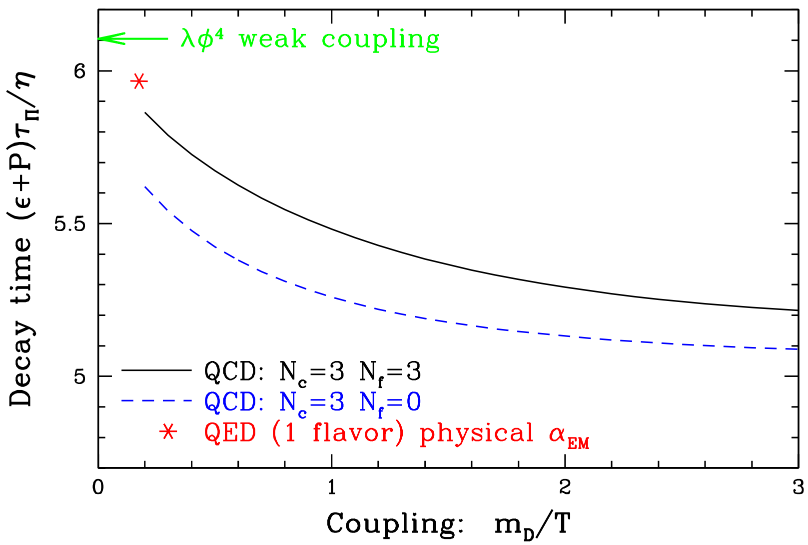
Preceding assumed that matrix element $|\mathcal{M}^2|$ is f independent.
In gauge and Yukawa theories, f enters $\mathcal{M}$ through screening!
Change in $f_0 \to f_0 + f_1$ changes screening, leading to correction to $|\mathcal{M}|^2$ linear in f_1.

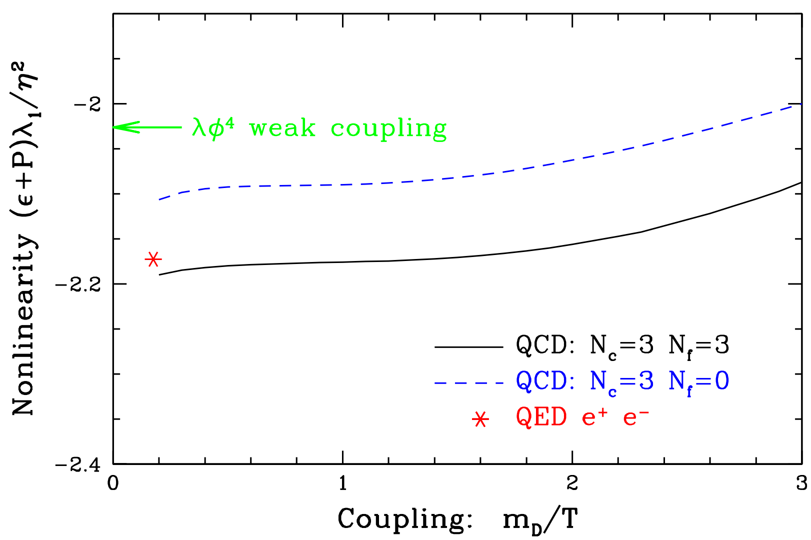
This is where things get hard.

So I won't tell you about it.

Results

$\lambda_3 = \kappa = 0$. $\lambda_2 = -2\eta\tau_\Pi$. τ_Π, λ_1 nontrivial:





Size of uncertainty is thinner than lines in plots!

Ratios are very stable with value of coupling.

QCD vs SYM comparison

Ratio	QCD value	SYM value
$\frac{\tau_\Pi(\epsilon+P)}{\eta}$	5 to 5.9	2.6137
$\frac{\lambda_1(\epsilon+P)}{\eta^2}$	-2 to -2.2	2
$\frac{\lambda_2(\epsilon+P)}{\eta^2}$	-10 to -11.8	-2.77
$\frac{\kappa(\epsilon+P)}{\eta^2}$	0	4
$\frac{\lambda_3(\epsilon+P)}{\eta^2}$	0	0

Good news: Not qualitatively different.

Bad news: "exact" kinetic theory relation $\lambda_2 = -2\eta\tau_\Pi$ not actually general.

Conclusions

- Hydro seems sensible framework in heavy ion coll.
- Shear viscosity should be quantified!
- Requires expansion to 2'nd order in gradients
- Calculation in pert. QCD is intricate.
- Ratios are relatively robust. But Pert Thy and SYM give rather different predictions in detail.

Limitation of kinetic theory method?

How does one compute non-kinetic corrections?