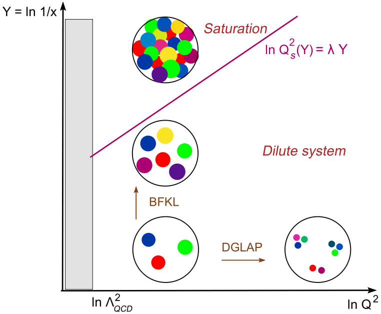
Impact of running coupling and higher order corrections on gluon saturation

Guillaume Beuf

Brookhaven National Laboratory

DIS 2011, Newport News, April 14

Kinematical regimes in DIS

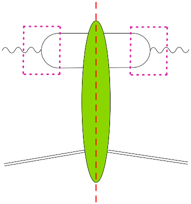


Saturation scale $Q_s(Y)$.

For $Q > Q_s(Y)$: dilute regime.

For $Q < Q_s(Y)$: dense regime, with a breakdown of the collinear factorization.

Dipole factorization of DIS



$$\sigma_{T,L}^{\gamma^* p}(Y, Q^2) = 2 \int d^2\mathbf{r} \sum_f \int_0^1 \mathrm{d}z |\Psi_{T,L}^f(\mathbf{r}, z, Q^2)|^2 \int d^2\mathbf{b}\; N(\mathbf{x}, \mathbf{y}, Y)$$

Virtual photon wave-function (known): $\Psi_{T,L}^f(\mathbf{r}, z, Q^2)$.

Dipole-target elastic scattering amplitude: $N(\mathbf{x}, \mathbf{y}, Y)$,

with a dipole of size $\mathbf{x} - \mathbf{y} = \mathbf{r}$ and impact parameter $\mathbf{b} = (\mathbf{x} + \mathbf{y})/2$.

Nikolaev, Zakharov (1991)

Balitsky Kovchegov (BK) equation at LO

$$\partial_Y N(\mathbf{x},\mathbf{y},Y) \;=\; \bar{\alpha}\int \frac{d^2\mathbf{z}}{2\pi}\,\frac{(\mathbf{x}-\mathbf{y})^2}{(\mathbf{x}-\mathbf{z})^2(\mathbf{z}-\mathbf{y})^2}\,\Big[N(\mathbf{x},\mathbf{z},Y)$$
$$+N(\mathbf{z},\mathbf{y},Y) \;-\; N(\mathbf{x},\mathbf{y},Y) - N(\mathbf{x},\mathbf{z},Y)\,N(\mathbf{z},\mathbf{y},Y)\Big]$$

Balitsky (1996), Kovchegov (1999-2000)

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Balitsky (1996), Kovchegov (1999-2000)

Approximation: impact parameter independence:
$N(\mathbf{x},\mathbf{y},Y) \simeq N(|\mathbf{r}|,Y)$,

with the prescription $\int d^2\mathbf{b}\; N(\mathbf{x},\mathbf{y},Y) \simeq \pi\, R^2_{target}\; N(|\mathbf{r}|,Y)$.

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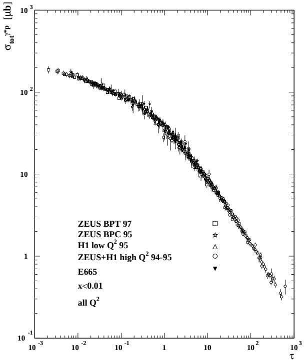
However there are issues with this assumption.
Berger, Staśto (2010)
Talk by J. Berger

Geometric scaling of the data



From the data: $\sigma_{tot}^{\gamma^* p}(Y, Q^2) = \sigma_{tot}^{\gamma^* p}(Q^2/Q_s^2(Y))$
with $Q_s^2(Y) \propto e^{\lambda Y}$ and $\lambda \simeq 0.2$ or 0.3.

Staśto, Golec-Biernat, Kwieciński (2001)

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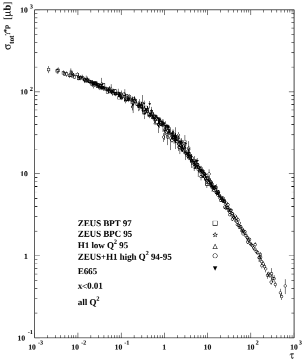

Or, thanks to the dipole factorization:

$$N(\mathbf{r}, Y) = N(\mathbf{r}^2\, Q_s^2(Y))$$

Traveling waves

Apart from numerical simulations, one of the most powerful tools to study the solutions of the BK equation is the traveling wave method.

In general: Ebert, van Saarloos (2000)

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It allows to

- Obtain the exact large Y behavior of the solutions of BK.
- Understand the outcome of numerical simulations.
- Obtain suitable informations to be put in phenomenological models. Iancu, Itakura, Munier (2004).

Formal BK equation

At leading logarithmic (LL) accuracy, the BK equation writes formally as

$$\partial_Y N(L,Y) \quad = \quad \bar{\alpha}\left\{\chi(-\partial_L)N(L,Y) - \text{Nonlinear damping}\right\}$$

with the BFKL eigenvalue $\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)$ and $L \sim -\log(r^2)$

Properties of saturation equations

The BK equation has the following properties:

1. Exponential growth and diffusion in the dilute regime $N(L, Y) \ll 1$.
2. Nonlinear damping in the dense regime $N(L, Y) \simeq 1$.
3. Family of uniformly translated wave-front (or exact scaling) solutions $N(L, Y) = N_\gamma(L - v\bar{\alpha}Y)$, which have
   - a tail in the UV of the type $N_\gamma(s) \propto e^{-\gamma s}$
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Theorem:
For the equations with such properties, all solutions with steep enough initial conditions converge to the uniformly translated wave-front solution of minimal velocity $v_c = v(\gamma_c)$, in an universal way.
Bramson (1983)

Universality of traveling wave solutions

In QCD: color transparency $\Rightarrow$ steep enough initial conditions $\Rightarrow$ all physically relevant solutions have a universal asymptotic behavior.

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$\longrightarrow$ Allows to better establish and extend previous analytical results from:
Gribov, Levin, Ryskin (1983)
Iancu, Itakura, McLerran (2002)
Mueller, Triantafyllopoulos (2002)

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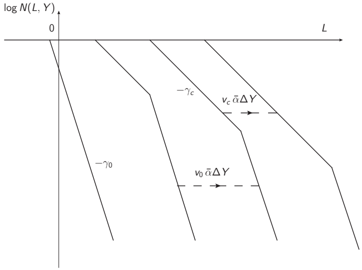

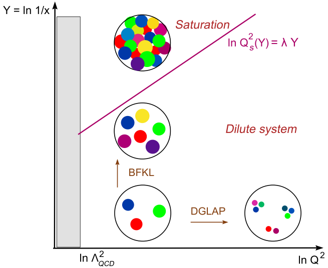

Three different regimes for the solutions:
the saturated regime, the universal dilute regime (or geometric scaling window), and the initial condition dominated dilute regime, which is progressively invaded by the universal one.

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Outcome: position of the wave-front / saturation scale:

$$L_s(Y) \equiv \log\left(\frac{Q_s(Y)^2}{{Q_0}^2}\right) = v_c\,\bar{\alpha}\,Y - \frac{3}{2\gamma_c}\log(\bar{\alpha}\,Y) + \mathrm{Const.} - \frac{3}{\gamma_c^2}\sqrt{\frac{2\pi}{\chi''(\gamma_c)\,\bar{\alpha}\,Y}} + \mathcal{O}\left(\frac{1}{\bar{\alpha}\,Y}\right)$$

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Remark: At fixed coupling the nuclear enhancement $Q_{s,A}^2 = A^{1/3}\,Q_{s,p}^2$ is introduced by the initial condition and left intact by the evolution, in the Const. term.

From fixed to running coupling

$$\partial_Y N(L,Y) \;=\; \bar{\alpha}\left\{\chi(-\partial_L)N(L,Y) - \text{Nonlinear damping}\right\}$$

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Parent dipole prescription for the running coupling:

$$\bar{\alpha} \mapsto \frac{1}{bL}\,, \quad \text{with} \quad b = \frac{11N_c - 2N_f}{12N_c}$$

$$\text{and} \quad L = -\log\left(\frac{r^2{\Lambda_{QCD}}^2}{4}\right)$$

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$\longrightarrow$ Also allows to address the case of different running running coupling prescription, like Balitsky's smaller dipole size prescription.

Properties of the equations with running coupling.

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$\Rightarrow$ A universal wave-front can form asymptotically.

Saturation scale with running coupling

Using the same method as in the fixed coupling case, the universal asymptotic solution $N(L, Y)$ with running coupling has been calculated in the relevant regime $L, Y \to \infty$ with $s(L, Y)/L \to 0$.

Then the saturation scale $Q_s(Y)$ is defined and calculated from:

$$N(L_s(Y), Y) = \kappa\,, \quad \text{with} \quad L_s(Y) \equiv \log \frac{Q_s^2(Y)}{\Lambda_{QCD}^2}$$

G.B., arXiv:1008.0498
G.B., *in preparation.*

Universal part of the saturation scale

$$
\begin{aligned}
L_s(Y) &= \log \frac{Q_s^2(Y)}{\Lambda_{QCD}^2} \\
&= \left(\frac{2v_c Y}{b}\right)^{1/2} + \frac{3\xi_1}{4}\left(D^2\,\frac{2v_c Y}{b}\right)^{1/6} \\
&\quad + c_0 + c_{-1/6}\left(D^2\,\frac{2v_c Y}{b}\right)^{-1/6} + c_{-1/3}\left(D^2\,\frac{2v_c Y}{b}\right)^{-1/3} \\
&\quad + \mathcal{O}\left(Y^{-1/2}\right)\,,
\end{aligned}
$$

with 3 new universal coefficients c_0, $c_{-1/6}$ and $c_{-1/3}$.

Notations

$$\gamma_c : \text{ solution of } \quad \chi(\gamma_c) = \gamma_c\, \chi'(\gamma_c)$$

$$v_c = \frac{\chi(\gamma_c)}{\gamma_c} \quad \text{and} \quad D = \frac{\chi''(\gamma_c)}{2\,\chi(\gamma_c)}$$

$\xi_1 \simeq -2.338$: rightmost zero of the Airy function.

Higher derivatives of the LL BFKL eigenvalue parameterized by:

$$K_n = \frac{2\,\chi^{(n)}(\gamma_c)}{n!\,\chi''(\gamma_c)} \quad \text{for} \quad n \geq 3$$

NLL contributions parameterized by:

$$N_n = \frac{\chi_{NLL}^{(n)}(\gamma_c)}{n!\,\chi(\gamma_c)\, b} \quad \text{for} \quad n \geq 0$$

Universal part of the saturation scale

$$c_0 = \Sigma - \frac{1}{\gamma_c} - \frac{K_3}{2} + N_0$$

$$c_{-1/6} = {\xi_1}^2 \left[\frac{1}{72\,\gamma_c^2} - \frac{D}{32} + \frac{K_3}{4\gamma_c} - \frac{3}{8}\,{K_3}^2 + \frac{3}{10}\,K_4 \right]$$

$$\begin{aligned} c_{-1/3} \quad = \quad & \xi_1 \left\{ \frac{\Sigma^2}{6\gamma_c} + \left[\frac{K_3}{2\gamma_c} - \frac{1}{3\gamma_c^2} \right] \Sigma + \frac{2}{27\gamma_c^3} + \frac{D}{3\gamma_c} \right. \\ & + K_3 \left[-\frac{2}{3\gamma_c^2} + \frac{3K_3}{4\gamma_c} - \frac{11{K_3}^2}{4} \right] + K_4 \left[-\frac{1}{\gamma_c} + 6K_3 \right] - 3K_5 \\ & \left. + \frac{1}{2\gamma_c^2} \left[1 + 3\gamma_c K_3 \right] \left[N_0 - \gamma_c N_1 \right] + N_2 - D N_0 \right\} \end{aligned}$$

Remarks: higher order contributions

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- Renormalization scheme independant result for $Q_s(Y)$.

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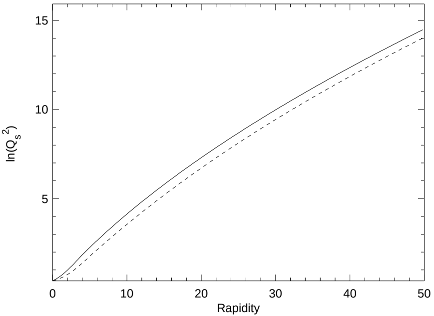

Berger, Staśto (2010)

Solid line: LO kernel with running coupling.

Dashed line: with in addition all order resummation of kinematical effects.

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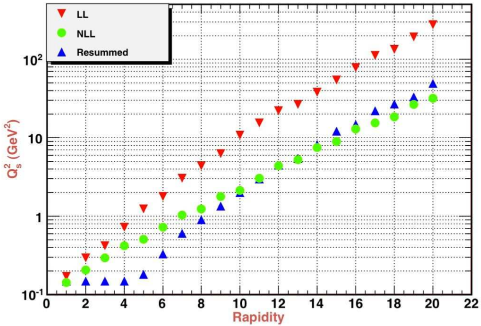

Avsar, Staśto, Triantafyllopoulos, Zaslavsky (work in progress)

Talk by E. Avsar

Remarks: suppression of initial condition dependance

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However, $Y^{-1/2}$ decreases quite slowly, in practice...

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- Effects of NLL and higher orders seen in the numerics understood qualitatively from analytical calculations.
  $\longrightarrow$ More quantitative comparisons?
- Suppression of the nuclear enhancement of Q_s at asymptotic rapidity.
  $\longrightarrow$ Slower A-dependance at low x at the EIC or LHeC?

Backup slides

Scaling variables and running coupling

Examples: there is one family of approximate scaling variables

$$s(L, Y) = \frac{L}{2\delta}\left[1-\left(\frac{2vY}{bL^2}\right)^{\delta}\right]$$

for each $\delta > 0$.

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For generic δ: Critical solution with same parameters $v = v_c$ and $\gamma = \gamma_c$ as as at fixed coupling.

- Particular case $\delta = 1/2$: *geometric scaling* variable:

$$s(L,Y) = L - \sqrt{\frac{2vY}{b}} \simeq -\log\left(\frac{r^2 Q_s^2(Y)}{4}\right)$$

$$\text{with} \quad Q_s^2(Y) \sim {\Lambda_{QCD}}^2 \exp\left(\sqrt{2vY/b}\right)$$

Scaling variables and running coupling

Approximate scaling variables for generic solutions with steep initial conditions:
$\longrightarrow$ has to include relaxation to the scaling variable with minimal velocity v_c, via diffusion effects.

$$\begin{aligned} s(L,Y) &= \frac{L}{2\delta}\left[1-\left(\frac{2v_c Y}{bL^2}\right)^{\delta}\right]-\frac{3\xi_1}{4}(DL)^{1/3} \\ &\quad +\eta+\mu(DL)^{-1/3}+\nu(DL)^{-2/3}+\mathcal{O}\left(L^{-1}\right) \end{aligned}$$

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First diffusive correction: already known. ξ_1: zero of Airy function. $D=\frac{\chi''(\gamma_c)}{2\,\chi(\gamma_c)}$.
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Mueller, Triantafyllopoulos (2002); Munier, Peschanski (2004)
Other coefficients η, μ and ν: new results.

Calculation of the universal asymptotics

First step:

Solve the evolution equation in the regime $L \to \infty$ and $s = \mathcal{O}(1)$, roughly parallel to lines of constant density, assuming a low enough density to linearize the equation:

$$N(L, Y) = \mathcal{A}\, e^{-\gamma_c s} \left[s + \frac{1}{\gamma_c} + (DL)^{-2/3}\, \phi_2(s) + (DL)^{-1}\, \phi_3(s) + \mathcal{O}\left(L^{-4/3} \right) \right]$$

with polynomial functions $\phi_2(s)$, $\phi_3(s)$, ...

Calculation of the universal asymptotics

Second step:

Solve the evolution equation in the regime $L \to \infty$ and $s = \mathcal{O}(L^{1/3})$, with $z = \frac{s}{(DL)^{1/3}} + \xi_1 = \mathcal{O}(1)$, were one follow the spreading of the universal solution into the dilute tail :

$$N(L,Y) \;=\; \mathcal{A}\, e^{-\gamma_c s}\Bigg[(DL)^{1/3}\,\frac{\mathrm{Ai}(z)}{\mathrm{Ai}'(\xi_1)} + G_0(z) + (DL)^{-1/3}\, G_1(z) + (DL)^{-2/3}\, G_2(z) + \mathcal{O}\left(L^{-1}\right)\Bigg]$$

where the $G_n(z)$ are Airy functions times polynomials.

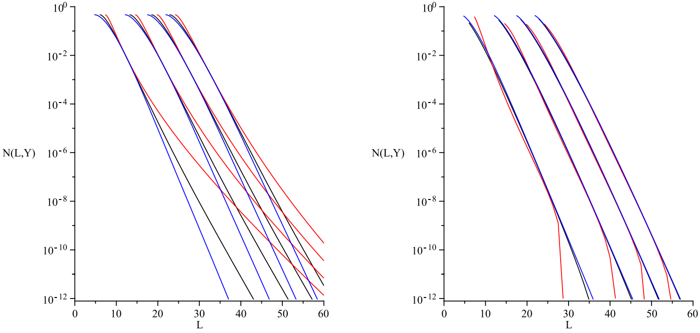
Calculation of the universal asymptotics

Third step:

- Match the two asymptotic expansions.
- Impose the causality requirements $G_n(z) \to 0$ for $z \to \infty$.
- Check that $N(L, Y)$ starts to saturate at a constant height.

$\Rightarrow$ At the order considered, all parameters and integration constants are determined, except the normalization constant $\mathcal{A}$ of $N(L, Y)$, which is nevertheless independent of initial conditions.

Dependance on the chosen scaling variable



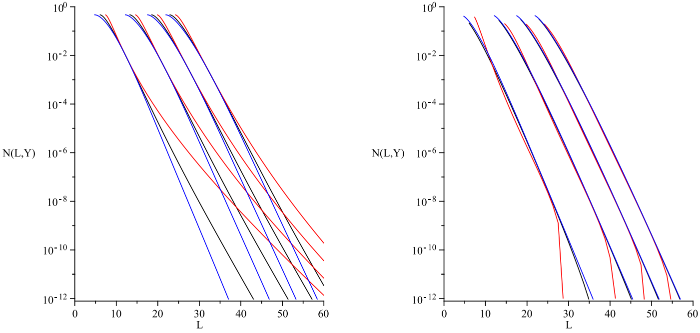
Asymptotic solution of the BK equation with running coupling (no NLL kernel). $Y = 10, 30, 50, 70$.

Blue: $\delta = 1/4$. Black: $\delta = 1/2$. Red: $\delta = 1$.

Left: with $G_0(z) = G_1(z) = G_2(z) = \eta = \mu = \nu = 0$.

Right: with all the orders calculated.

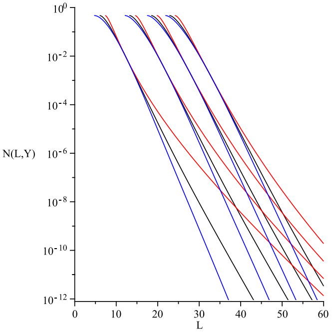
Dependance on the chosen scaling variable



Dependance on δ weaker and weaker when increasing the accuracy of the calculations.

$\rightarrow$ Exact solution independent on the choice of scaling variable $s(L, Y)$.

Dependance on the chosen scaling variable



→ Shows without bias the validity of an effective approximate geometric scaling in the running coupling case.

Extraction of the saturation scale

The saturation scale $Q_s(Y)$ is defined by a level line:

$$N(L_s(Y), Y) = \kappa\,, \quad \text{with} \quad L_s(Y) \equiv \log \frac{Q_s^2(Y)}{\Lambda_{QCD}^2}$$

From the result

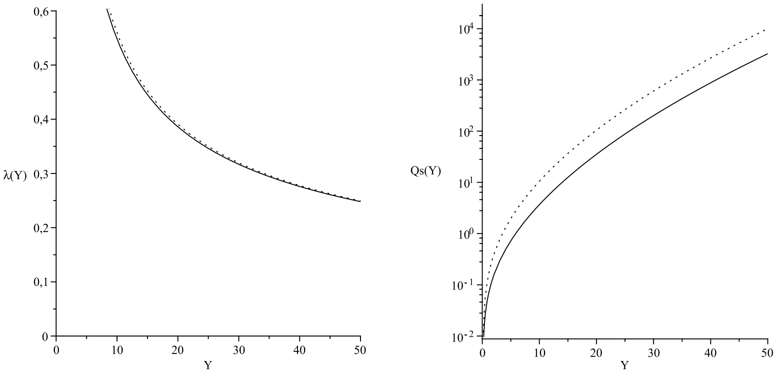
$$N(L, Y) = \mathcal{A}\, e^{-\gamma_c s} \left[s + \frac{1}{\gamma_c} + (DL)^{-2/3}\, \phi_2(s) + \mathcal{O}\left(L^{-1}\right) \right],$$

$$\text{one gets} \quad s(L_s(Y), Y) = \Sigma - \frac{1}{\gamma_c} + (D\, L_s(Y))^{-2/3}\, \frac{\phi_2(\Sigma - 1/\gamma_c)}{\gamma_c \Sigma - 1}\,,$$

$$\text{where} \quad \Sigma \equiv -\frac{1}{\gamma_c}\, \mathrm{W}_{-1}\left(-\frac{\gamma_c\, \kappa}{\mathcal{A}\, e} \right)\,,$$

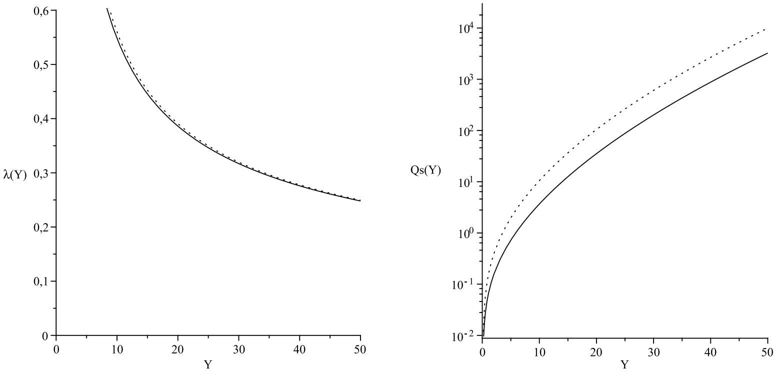
and $\mathrm{W}_{-1}(x)$ is the -1 branch of the Lambert function.

$Q_s(Y)$: Parent dipole *vs.* Balitsky's prescription



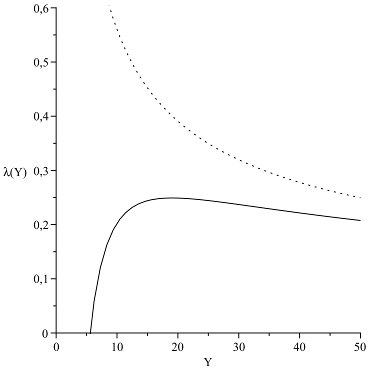
Solid line: with Balitsky's prescription.
Dotted line: with the parent dipole prescription.

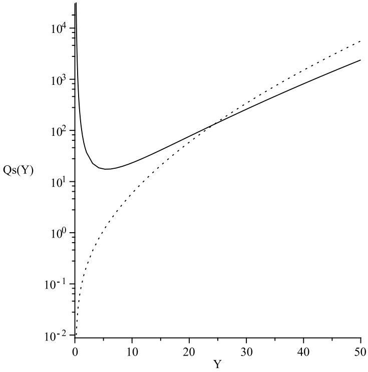
$Q_s(Y)$: Parent dipole *vs.* Balitsky's prescription



$\Rightarrow$ Running coupling prescription: important for the normalization of $Q_s(Y)$ but not for its evolution, at large Y.

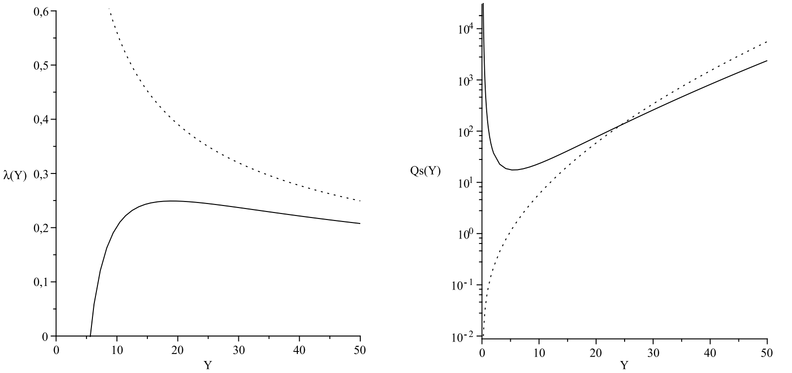
$Q_s(Y)$ with the full NLL BFKL kernel





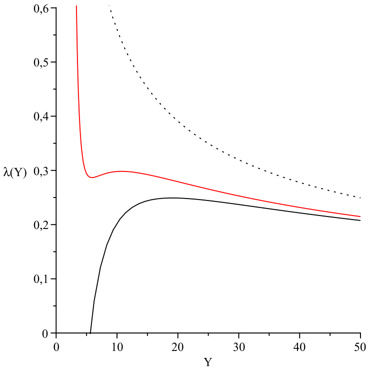
Solid line: with the full NLL BFKL kernel.
Dotted line: with no NLL contribution besides running coupling with parent gluon $k_\perp$.

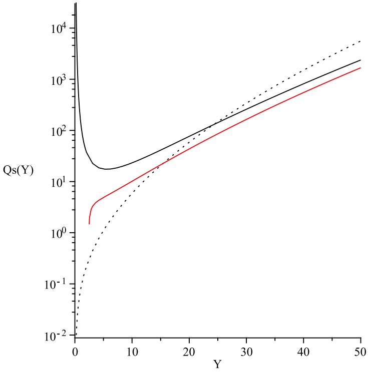
$Q_s(Y)$ with the full NLL BFKL kernel



$\Rightarrow$ Full NLL effects stabilize $\lambda(Y)$ in the phenomenologically preferred range 0.2 - 0.3.

$Q_s(Y)$ with the full NLL BFKL kernel





Test of subleading initial conditions:

Black: universal terms only.

Red: with a shift $Y \mapsto Y - 2.5$ in the leading term only.