TMD Factorization, Factorization Breaking and Evolution

Ted C. Rogers

*Vrije Universiteit Amsterdam*

Based on:

S.M. Aybat, TCR (2011)

TCR, P.J. Mulders (2010)

See also, *Foundations of Perturbative QCD*, J.C. Collins.

*(available May 2011)*

DIS 2011 April 13, 2011
Status:

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
Status:

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
  - Much progress! (see pre-DIS talks)
**Status:**

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
  - *Much progress! (see pre-DIS talks)*

- **Related:** Factorization Theorems:

  *Definitions dictated by requirements for factorization!*
**Status:**

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
  - *Much progress! (see pre-DIS talks)*

- **Related:** Factorization Theorems:
  - Semi-Inclusive deep inelastic scattering.
  - Drell-Yan.
  - $e^+/e^-$ annihilation.
  - $p + p \rightarrow h_1 + h_2 + X$
Status:

• Complicated issues involved in defining TMDs.
  – Divergences.
  – Wilson lines / gauge links.
  – Universality / Non-universality.
  – *Much progress! (see pre-DIS talks)*

• Related: Factorization Theorems:
  – Semi-Inclusive deep inelastic scattering. ✓
  – Drell-Yan. ✓
  – $e^+e^-$ annihilation. ✓
  – $p + p \rightarrow h_1 + h_2 + X$

Watch out for sign flips!
Status:

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
  - Much progress! (see pre-DIS talks)

- Related: Factorization Theorems:
  - Semi-Inclusive deep inelastic scattering. ✓
  - Drell-Yan. ✓
  - $e^+e^\mp$ annihilation. ✓
  - $p + p \rightarrow h_1 + h_2 + X$ ✗

Watch out for sign flips!
**Status:**

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
  - *Much progress! (see pre-DIS talks)*

- Related: Factorization and Universality:
  - Semi-Inclusive deep inelastic scattering.
  - Drell-Yan.
  - $e^+/e^-$ annihilation.
  - $p + p \rightarrow h_1 + h_2 + X$ !!

- Related: Implementation and Evolution.
**Status:**

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
  - *Much progress! (see pre-DIS talks)*

- **Related:** Factorization and Universality:
  - Semi-Inclusive deep inelastic scattering.
  - Drell-Yan.
  - $e^+/e^-$ annihilation.
  - $p + p \rightarrow h_1 + h_2 + X$ !

- **Related:** Implementation and Evolution.
  - Existing fixed-scale fits with no evolution.
**TMD-Factorization**

- TMD Parton model intuition (Drell-Yan):

\[
W^{\mu\nu} = \sum_f |\mathcal{H}_f(Q)^2|^{\mu\nu} \int d^2k_{1T} d^2k_{2T} F_f/P_2(x_1, k_{1T}) F_{\bar{f}}/P_2(x_2, k_{2T}) \times \delta^{(2)}(k_{1T} + k_{2T} - q_T)
\]

*Leading order hard part*  
*No evolution*

**Generalized Parton Model**
**Status:**

- Complicated issues involved in defining TMDs.
  - Divergences.
  - Wilson lines / gauge links.
  - Universality / Non-universality.
  - *Much progress! (see pre-DIS talks)*

- **Related:** Factorization and Universality:
  - Semi-Inclusive deep inelastic scattering.
  - Drell-Yan.
  - $e^+e^-$ annihilation.

- **Related:** Implementation and Evolution.
  - Existing fixed-scale with no evolution.
  - Existing “old fashion” implementations of Collins-Soper-Sterman formalism.
Evolved Cross Section:

- Contrast: Typical appearance of Collins-Soper-Sterman formalism: 

\[ d\sigma \sim \int d^2 b \ e^{-ib \cdot q_T} \]

\[
\int_{x_1}^{1} d\hat{x}_1 \frac{\tilde{C}_{f/j}(x_1/\hat{x}_1, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P_1}(\hat{x}_1, \mu_b)}{\hat{x}_1} \\
\int_{x_2}^{1} d\hat{x}_2 \frac{\tilde{C}_{f/j}(x_2/\hat{x}_2, b_*; \mu_b^2, \mu_b, g(\mu_b)) f_{j/P_2}(\hat{x}_2, \mu_b)}{\hat{x}_2} \\
\exp \left[ \int_{1/b^2}^{Q^2} \frac{d\mu'^2}{\mu'^2} \left\{ A(\alpha_s(\mu')) \ln \frac{Q^2}{\mu'^2} + B(\alpha_s(\mu')) \right\} \right] \\
\exp \left[ -g_k(b) \ln \frac{Q^2}{Q_0^2} - g_1(x_1, b) - g_2(x_2, b) \right] \\
+ \text{Large } q_T \text{ term} \]
Status:

• Complicated issues involved in defining TMDs.
  – Divergences.
  – Wilson lines / gauge links.
  – Universality / Non-universality.
  – *Much progress! (see pre-DIS talks)*

• Related: Factorization and Universality:
  – Semi-Inclusive deep inelastic scattering.
  – Drell-Yan.
  – $e^+/e^-$ annihilation.
  – $p + p \rightarrow h_1 + h_2 + X$!!

• Related: Implementation and Evolution.
  – Existing fixed-scale with no evolution.
  – Existing “old fashion” implementations of Collins-Soper-Sterman formalism.

• How are these related? How to connect to phenomenology?
What is needed?

- TMD Parton model intuition (Drell-Yan):

\[ W^{\mu\nu} = \sum_f |H_f(Q)^2|^{\mu\nu} \int d^2k_{1T} d^2k_{2T} F_{f/P_2}(x_1, k_{1T}) F_{\bar{f}/P_2}(x_2, k_{2T}) \times \delta^{(2)}(k_{1T} + k_{2T} - q_T) \]

- Using newest definitions:

\[ W^{\mu\nu} = \sum_f |H_f(Q; \mu)^2|^{\mu\nu} \times \int d^2k_{1T} d^2k_{2T} F_{f/P_1}(x_1, k_{1T}; \mu; \zeta_1) F_{\bar{f}/P_2}(x_2, k_{2T}; \mu; \zeta_2) \times \delta^{(2)}(k_{1T} + k_{2T} - q_T) + Y(Q, q_T) + \mathcal{O}((\Lambda/Q)^a). \]
What is needed?

- TMD Parton model intuition (Drell-Yan):

\[ W^{\mu\nu} = \sum_{f} |\mathcal{H}_f(Q)^2|^{\mu\nu} \int d^2k_1T \, d^2k_2T \, F_{f/P_2}(x_1, k_{1T}) \, F_{\bar{f}/P_2}(x_2, k_{2T}) \times \]

\[ \times \delta^{(2)}(k_{1T} + k_{2T} - q_T) \]

- Using newest definitions:

\[ W^{\mu\nu} = \sum_{f} |\mathcal{H}_f(Q; \mu)^2|^{\mu\nu} \]

\[ \times \int d^2k_1T \, d^2k_2T \, F_{f/P_1}(x_1, k_{1T}; \mu; \zeta_1) \, F_{\bar{f}/P_2}(x_2, k_{2T}; \mu; \zeta_2) \times \]

\[ \times \delta^{(2)}(k_{1T} + k_{2T} - q_T) \]

\[ + Y(Q, q_T) + \mathcal{O}((\Lambda/Q)^a). \]
TMD PDF, Complete Definition:

\[ F_{f/P}(x, b; \mu; \zeta_F) = \]

"Unsubtracted"

Implements Subtractions/Cancellations

From *Foundations of Perturbative QCD*, J.C. Collins,
See also, Collins, TMD 2010 Trento Workshop
Our Strategy:

• Use evolution to extrapolate between existing fits, to build unified fits that include evolution.

  \[\text{(S.M. Aybat, TCR (2011))}\]

– PDFs:
  • Start with DY:
    \[\text{(Landry et al, (2003); Konychev, Nadolsky (2006)) (BLNY)}\]
  • Modify to match to SIDIS:
    \[\text{(Schweitzer, Teckentrup, Metz (2010)) (STM)}\]
  \[\text{(For details, see Aybat talk, pre-DIS meeting.)}\]

• Can supply explicit, evolved TMD PDF fit.
Evolving TMD PDFs

Up Quark TMD PDF, $x = 0.09$

$b_{T,\text{max}} = 0.5 \text{ GeV}^{-1}$

- $Q = \sqrt{2.4} \text{ GeV}$
- $Q = 5.0 \text{ GeV}$
- $Q = 91.19 \text{ GeV}$

(Schweitzer, Teckentrup, Metz (2010))
(SIDIS)

(Landry et al, (2003))
(Drell-Yan)
Evolving TMD PDFs

Up Quark TMD PDF, \( x = 0.09 \)

\[ F_{u/p}(x^{0.09}, k_T) \text{ (GeV}^2) \]

- **JLab Energies**
- **Tevatron Energies**

- \( b_{T,\text{max}} = 0.5 \text{ GeV}^{-1} \)
- \( Q = \sqrt{2.4} \text{ GeV} \)
- \( Q = 5.0 \text{ GeV} \)
- \( Q = 91.19 \text{ GeV} \)
Evolving TMD PDFs

Up Quark TMD PDF, $x = .09$, $Q = 91.19$ GeV

Gaussian fit good at small $k_T$. 

$F_{up}(x=0.09, k_T) (\text{GeV}^2)$

$k_T$ (GeV)

$b_{T,\text{max}} = .5 \text{ GeV}^{-1}$

Q = 91.19 GeV

Gaussian Fit
Unambiguous Hard Part

• Higher orders follow systematically from definitions:

\[ W^{\mu\nu} = |\mathcal{H}_f(Q; \mu/Q)^2|^{\mu\nu} F_{f/P_1} \otimes F_{f/P_2} \]

\[ |\mathcal{H}_f(Q; \mu/Q)^2|^{\mu\nu} = \frac{W^{\mu\nu}}{F_{f/P_1} \otimes F_{f/P_2}} \]
Unambiguous Hard Part

- **Definition:**

\[ {\mathcal H_f}(Q; \mu/Q)^2 |^{\mu\nu} = \frac{W^{\mu\nu}}{F_f/P_1 \otimes F_f/P_2} \]

- **Drell-Yan:** (MS)

\[ {\mathcal H_f}(Q; \mu/Q)^2 |^{\mu\nu} = e_f^2 |H_0^2|^{\mu\nu} \left( 1 + \frac{C_F \alpha_s}{\pi} \left[ \frac{3}{2} \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{1}{2} \ln^2 \left( \frac{Q^2}{\mu^2} \right) - 4 + \frac{\pi^2}{2} \right] \right) + O(\alpha_s^2) \]
Unambiguous Hard Part

- Definition:

\[ |\mathcal{H}_f(Q; \mu/Q)^2|^{\mu\nu} = \frac{W^{\mu\nu}}{F_f/P_1 \otimes F_f/P_2} \]

- Drell-Yan: (\text{MS})

\[ |\mathcal{H}_f(Q; \mu/Q)^2|^{\mu\nu} = e^2_f |H_0^2|^{\mu\nu} \left( 1 + \frac{C_F \alpha_s}{\pi} \left[ \frac{3}{2} \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{1}{2} \ln^2 \left( \frac{Q^2}{\mu^2} \right) - 4 + \frac{\pi^2}{2} \right] \right) + \mathcal{O}(\alpha_s^2) \]

- SIDIS

\[ |\mathcal{H}_f(Q; \mu/Q)^2|^{\mu\nu} = e^2_f |H_0^2|^{\mu\nu} \left( 1 + \frac{C_F \alpha_s}{\pi} \left[ \frac{3}{2} \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{1}{2} \ln^2 \left( \frac{Q^2}{\mu^2} \right) - 4 \right] \right) + \mathcal{O}(\alpha_s^2) \]
Unambiguous Hard Part

• Definition:

\[ H_f(Q; \mu/Q)^2 \big|_{\mu\nu} = \frac{W_{\mu\nu}}{F_f / P_1 \otimes F_f / P_2} \]

• Drell-Yan: (MS)

\[ H_f(Q; \mu/Q)^2 \big|_{\mu\nu} = \left( 1 + \frac{C_F \alpha_s}{\pi} \left[ \frac{3}{2} \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{1}{2} \ln^2 \left( \frac{Q^2}{\mu^2} \right) - 4 + \frac{\pi^2}{2} \right] \right) + O(\alpha_s^2) \]

• SIDIS

\[ H_f(Q; \mu/Q)^2 \big|_{\mu\nu} = \left( 1 + \frac{C_F \alpha_s}{\pi} \left[ \frac{3}{2} \ln \left( \frac{Q^2}{\mu^2} \right) - \frac{1}{2} \ln^2 \left( \frac{Q^2}{\mu^2} \right) - 4 \right] \right) + O(\alpha_s^2) \]
**Long-Term Goal:**

- Repository of new TMD fits with evolution.

- Based on well-understood operator definitions.
  - *Take Collins definitions.*
**Agenda:**

- Improve fits. Combine SIDIS, DY, e\(^+\)e\(^-\) in global fit. Extend to higher orders. Gaussian fits.

- Extend to polarization dependent functions (Sivers, Boer-Mulders, etc...).

- TMD gluon distribution.

- Factorization breaking??

- Updates to appear at: [https://projects.hepforge.org/tmd/](https://projects.hepforge.org/tmd/)
Thanks!
Backup Slides
**Understanding the Definition:**

- Start with only the hard part factorized:

  

  \[ d\sigma = \left| H \right|^2 \tilde{F}_1^{\text{unsub}}(y_1 - (-\infty)) \times \tilde{F}_2^{\text{unsub}}(\infty - y_2). \]

  

**Naïve Factorization:**
**Understanding the Definition:**

- Start with only the hard part factorized:

  **Naïve Factorization:**

  \[
  d\sigma = |\mathcal{H}|^2 \tilde{F}_1^{\text{unsub}}(y_1 - (-\infty)) \times \tilde{F}_2^{\text{unsub}}(+\infty - y_2).
  \]
**Understanding the Definition:**

- Start with only the hard part factorized:

\[
\frac{\hat{F}^{\text{unsub}}_1 (y_1 - (-\infty)) \times \hat{F}^{\text{unsub}}_2 (+\infty - y_2)}{\tilde{S}(+\infty, -\infty)}.
\]
Understanding the Definition:

- Start with only the hard part factorized:
  \[ d\sigma = |H|^2 \frac{\tilde{F}_{1\text{unsub}}(y_1 - (-\infty)) \times \tilde{F}_{2\text{unsub}}(+\infty - y_2)}{\tilde{S}(+\infty, -\infty)}. \]

- Separate soft part:
  \[ d\sigma = |H|^2 \frac{F_{1\text{unsub}}(y_1 - (-\infty))}{\sqrt{\tilde{S}(+\infty, -\infty)}} \times \frac{F_{2\text{unsub}}(+\infty - y_2)}{\sqrt{\tilde{S}(+\infty, -\infty)}}. \]
Understanding the Definition:

- Start with only the hard part factorized:
  \[
  d\sigma = |\mathcal{H}|^2 \frac{\tilde{F}^{\text{unsub}}_1(y_1 - (-\infty)) \times \tilde{F}^{\text{unsub}}_2(+\infty - y_2)}{\tilde{S}(+\infty, -\infty)}.
  \]

- Separate soft part:
  \[
  d\sigma = |\mathcal{H}|^2 \frac{F^{\text{unsub}}_1(y_1 - (-\infty))}{\sqrt{\tilde{S}(+\infty, -\infty)}} \times \frac{\tilde{F}^{\text{unsub}}_2(+\infty - y_2)}{\sqrt{\tilde{S}(+\infty, -\infty)}}.
  \]

- Multiply by:
  \[
  \frac{\sqrt{\tilde{S}(+\infty, y_s)} \tilde{S}(y_s, -\infty)}{\sqrt{\tilde{S}(+\infty, y_s)} \tilde{S}(y_s, -\infty)}
  \]
Understanding the Definition:

• Start with only the hard part factorized:

\[ d\sigma = |\mathcal{H}|^2 \frac{\tilde{F}_{\text{unsub}}^1(y_1 - (-\infty)) \times \tilde{F}_{\text{unsub}}^2(+\infty - y_2)}{\tilde{S}(+\infty, -\infty)}. \]

• Separate soft part:

\[ d\sigma = |\mathcal{H}|^2 \frac{\tilde{F}_{\text{unsub}}^1(y_1 - (-\infty)) \times \tilde{F}_{\text{unsub}}^2(+\infty - y_2)}{\sqrt{\tilde{S}(+\infty, -\infty)} \times \sqrt{\tilde{S}(+\infty, -\infty)}}. \]

• Multiply by:

\[ \sqrt{\tilde{S}(+\infty, y_s) \tilde{S}(y_s, -\infty)} \]
\[ \sqrt{\tilde{S}(+\infty, y_s) \tilde{S}(y_s, -\infty)} \]

• Rearrange factors: \( d\sigma = |\mathcal{H}|^2 \left\{ \frac{\tilde{F}_{\text{unsub}}^1(y_1 - (-\infty))}{\sqrt{\tilde{S}(+\infty, y_s) \tilde{S}(y_s, -\infty)}} \right\} \]
\[ \times \left\{ \frac{\tilde{F}_{\text{unsub}}^2(+\infty - y_2)}{\sqrt{\tilde{S}(+\infty, -\infty) \tilde{S}(+\infty, y_s)}} \right\} \]
Understanding the Definition:

- Start with only the hard part factorized:

  \[ d\sigma = |\mathcal{H}|^2 \frac{\tilde{F}_1^{\text{unsub}}(y_1 - (-\infty)) \times \tilde{F}_2^{\text{unsub}}(+\infty - y_2)}{\tilde{S}(+\infty, -\infty)}. \]

- Separate soft part:

  \[ d\sigma = |\mathcal{H}|^2 \frac{F_1^{\text{unsub}}(y_1 - (-\infty))}{\sqrt{\tilde{S}(+\infty, -\infty)}} \times \frac{\tilde{F}_2^{\text{unsub}}(+\infty - y_2)}{\sqrt{\tilde{S}(+\infty, -\infty)}}. \]

- Multiply by:

  \[ \sqrt{\tilde{S}(+\infty, y_s) \tilde{S}(y_s, -\infty)} \]
  \[ \sqrt{\tilde{S}(+\infty, y_s) \tilde{S}(y_s, -\infty)} \]

- Rearrange factors: \( d\sigma = |\mathcal{H}|^2 \left\{ \frac{F_1^{\text{unsub}}(y_1 - (-\infty)) \sqrt{\tilde{S}(+\infty, y_s)}}{\tilde{S}(+\infty, -\infty) \tilde{S}(y_s, -\infty)} \right\} \]

  \[ \times \left\{ \tilde{F}_2^{\text{unsub}}(+\infty - y_2) \sqrt{\frac{\tilde{S}(y_s, -\infty)}{\tilde{S}(+\infty, -\infty) \tilde{S}(+\infty, y_s)}} \right\} \]

  \textit{Separately Well-defined}
Evolution

- Collins-Soper Equation:

\[
\frac{\partial \ln \tilde{F}(x, b_T, \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)
\]

\[
\tilde{K}(b_T; \mu) = \frac{1}{2 \partial y_n} \ln \frac{\tilde{S}(b_T; y_n, -\infty)}{\tilde{S}(b_T; +\infty, y_n)}
\]

- RG:

\[
\frac{d \tilde{K}}{d \ln \mu} = -\gamma_K(g(\mu))
\]

\[
\frac{d \ln \tilde{F}(x, b_T; \mu, \zeta)}{d \ln \mu} = -\gamma_F(g(\mu); \zeta/\mu^2)
\]

Perturbatively calculable, from definitions

Perturbatively calculable from definition at small b.
**TMD-Factorization**

- TMD-factorization with consistent definitions:

\[
W^{\mu\nu} = \sum_f |\mathcal{H}_f(Q; \mu)|^2 |^{\mu\nu} \\
\times \int d^2k_{1T} d^2k_{2T} \left( F_{f/P_1}(x_1, k_{1T}; \mu; \zeta_1) F_{\bar{f}/P_2}(x_2, k_{2T}; \mu; \zeta_2) \right) \\
\times \delta^{(2)}(k_{1T} + k_{2T} - q_T) \\
\times Y(Q, q_T) + \mathcal{O}((\Lambda/Q)^a).
\]
Implementing Evolution

• After evolution:

\[
\tilde{F}_{f/H}(x, b_T, \mu, \zeta) = \sum_j \int_x^1 \frac{d\hat{x}}{\hat{x}} \tilde{C}_{f/j}(x/\hat{x}, b_*; \mu_b, g(\mu_b)) f_{j/H}(x, \mu_b) \times \left\{ \begin{array}{l}
\times \exp \left\{ \ln \frac{\sqrt{\zeta}}{\mu_b} \tilde{K}(b_*; \mu_b) + \int_{\mu_b}^{\mu} \frac{d\mu'}{\mu'} \left[ \gamma_F(g(\mu'); 1) - \ln \frac{\sqrt{\zeta}}{\mu'} \gamma_K(g(\mu')) \right] \right\} \times \\
\times \exp \left\{ g_{j/H}(x, b_T) + g_K(b_T) \ln \frac{\sqrt{\zeta}}{Q_0} \right\}
\end{array} \right. \}
\]

\[
b_*(b_T) \equiv \frac{b_T}{\sqrt{1 + b_T^2 / b_{max}^2}} \qquad \mu_b(b_T) \sim 1/b_*
\]

CSS matching procedure
Up Quark TMD PDF, $x = 0.09$, $Q = 91.19$ GeV

$F_{up}^{p}(x=0.09,k_t)$ (GeV$^{-2}$)

$b_{T,\text{max}} = 0.5$ GeV$^{-1}$

- Red: $Q = 91.19$ GeV
- Blue: $A = f(x)$
- Purple: $B = 1$

$b_{T,\text{max}} = 1.5$ GeV$^{-1}$